Problems from List 8

$\boxed{6\ (iii)}$ Prove that any $f: \mathbb{B}^n \to \mathbb{B}^n$ continuous has a fixed point, assuming this is true for smooth maps. ($\mathbb{B}^n$ = closed unit ball in $\mathbb{R}^n$).

Pf. By Stone-Weierstrass, for any $\varepsilon > 0$ we may find $g: \mathbb{B}^n \to \mathbb{R}^n$ smooth so that $\| g(x) - f(x) \| < \varepsilon \quad \forall x \in \mathbb{B}^n$ (each component g_i of g is the restriction of a polynomial $\mathbb{R}^n \to \mathbb{R}$ approximating f_i on $\mathbb{B}^n$.)

Assume (by contr.) $f(x) \neq x \quad \forall x \in \mathbb{B}^n$. Then $\| f(x) - x \| > \delta > 0 \quad \forall x \in \mathbb{B}^n$ (for some $\delta > 0$, by continuity). Note $\| g(x) \| \le \| g(x) - f(x) \| + \| f(x) \| < 1 + \varepsilon$

So let $\hat{g}: \mathbb{B}^n \to \mathbb{B}^n$ be defined by $\hat{g}(x) = \dfrac{g(x)}{1+\varepsilon}$.

$\hat{g}$ is smooth, so it has a fixed pt. in $\mathbb{B}^n$. Also:

$$\| \hat{g}(x) - g(x) \| = \| g(x) \| \left(1 - \frac{1}{1+\varepsilon} \right) < (1+\varepsilon) \cdot \frac{\varepsilon}{1+\varepsilon} = \varepsilon .$$

Thus $\| f(x) - \hat{g}(x) \| < 2\varepsilon$. But then; for all $x \in \mathbb{B}^n$:

$$\| \hat{g}(x) - x \| \geq \| f(x) - x \| - \| f(x) - \hat{g}(x) \| > \delta - 2\varepsilon .$$

Contradiction if we choose $\varepsilon < \delta/2$.

$\boxed{8}$ $f: X \to Y$ smooth (X compact, Y connected) $\dim(X) = \dim(Y)$.

Prove that ~~card $f^{-1}(q)$~~ $\deg_2(f; q)$ is constant mod 2 (over all $q \in Y$).

Claim For each $N \in \mathbb{N} = \{0, 1, 2, \dots\}$ the set $C_N = \{ y \in Y \mid \deg_2(f, y) \equiv N \bmod 2 \}$ is open in Y.

Pf. Let $y_0 \in C_N$.

$\exists\, g: X \to Y$ C^1-close to f, $g \simeq f$ (homotopic) s.t. y_0 is a reg. value for g (by the transversality homotopy theorem). Thus card $g^{-1}(y_0) = M$ ($M \equiv N\ (2)$)

Let $g^{-1}(y_0) = \{x_1, \dots, x_M\}$. $\exists\, U_i \subset X$ nbd of x_i (all the U_i disjoint) and $V \subset Y$ nbd of y_0 s.t. $g: U_i \to V$ is a diffeo. In part. card $g^{-1}(z) = M$, for all $z \in V$, so $\deg_2(g; z) = N \bmod 2$. By homotopy invariance, $\deg_2(f; z) = N \bmod 2 \quad \forall z \in V$. This proves the claim.

Since $Y = C_0 \sqcup C_1$ (disjoint), both open, exactly one of C_0, C_1 is nonempty.

Thus either $\deg_2(f, y) \equiv 0 \ \forall y$ or $\deg_2(f, y) \equiv 1 \ \forall y \pmod 2$

Remark The statement in problem 8 is false: although $\deg_2(f;y)$ is constant over $y \in Y$, $\text{card}\, f^{-1}(y)$ is not constant mod 2.

Let X be the simple closed curve shown, Y the circle, $f: X \to Y$ the radial projection

$\text{card}\, f^{-1}(q_1) = 3$

$\text{card}\, f^{-1}(q_2) = 1$

but $\text{card}\, f^{-1}(q_3) = 2$

(where the ray from q_3 is tangent to X at one point of ~~each~~ $f^{-1}(q_3)$.

[2] $U \subset \mathbb{R}^k$, $V \subset \mathbb{R}^k_+$ open nbds of $0 \in \mathbb{R}^k$ (so V contains points of $\mathbb{R}^{k-1} \times \{0\} = \partial \mathbb{R}^k_+$ (in $\mathbb{R}^k$)).

By contr., assume $\exists\ f: U \to V$ diffeo. Let $p = f^{-1}(0) \in U$.

Then $\exists\ W_p \subset U$ nbd of p so that $f: W_p \to f(W_p) \subset V$ is a diffeo, and $f(W_p)$ is a nbd. of 0 in $\mathbb{R}^k_+$. In particular $df(p) \in \mathcal{L}(\mathbb{R}^k)$ is an isomorphism, and by the IFT for $f: W_p \to \mathbb{R}^k$, f is a diffeo from $W'_p \subset W_p$ to V', a nbd of 0 in $\mathbb{R}^k$, which in particular contains points in $\mathbb{R}^k \setminus \mathbb{R}^k_+ = \{x \in \mathbb{R}^k ; x_k < 0\}$. Contradicts $f(W_p) \subset \mathbb{R}^k_+$.

[3] By (2): $g = \partial f$ maps ∂X to ∂Y bijectively and continuously, and g^{-1} maps ∂Y to ∂X.

If $p \in \partial X$ $\exists$ nbd V of p in X s.t. $f: V \to f(V)$ diffeomorphically, so

$\partial f|_V : V \cap \partial X \to f(V) \cap \partial Y$ is a diff'ble homeomorphism (w.r.t. induced top.) w/ diff'ble inverse; i.e. a diffeomorphism.

$\boxed{12}$ (i) Denote by $\exp:\mathbb{R}\to S^1$ the covering map $\exp(t)=e^{it}$

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ {\scriptstyle\exp}\downarrow & & \downarrow{\scriptstyle\exp} \\ S^1 & \xrightarrow{f} & S^1 \end{array}$$

By the lifting theorem for coverings, there is a unique (smooth) map $g:\mathbb{R}\to\mathbb{R}$ lifting $f\circ\exp:\mathbb{R}\to S^1$ over exp,

(so $\exp\circ g = f\circ\exp$) with $g(0)=t_0$ (where $e^{it_0}=f(1)$)

Then $f(e^{it}) = e^{ig(t)}$ implies $g(2\pi)=g(0)+2\pi q$ for some $q\in\mathbb{Z}$.

Indeed $g(t+2\pi)=g(t)+2\pi q$ $\forall t\in\mathbb{R}$ (both are lifts w/ the value $t_0+2\pi q$ at $t=0$)

(ii) We replace f by a map homotopic to f to compute the degree.

Let $g_s(t) = (1-s)\,g(t)+s\,h(t)$, $\quad h(t) = g(0)+\frac{t}{2\pi}\left(g(2\pi)-g(0)\right)$

Note $g_s(2\pi)-g_s(0) = 2\pi q$ $\forall s\in[0,1]$ $\quad = g(0)+tq$.

So $f_s(e^{it}) = e^{ig_s(t)}$ defines $f_s:S^1\to S^1$ smooth, homotopy from f to $e^{ih(t)} = f(1)\,e^{itq}$ (call this $L:S^1\to S^1$).

Note every point $\omega\in S^1$ is a reg. value for L. Say $\omega=f(1)$.

Then $L^{-1}(\omega) = \exp\{t \mid \omega e^{itq}=\omega\} = \exp\{t \mid e^{itq}=1\}$

$= \exp\{0, \frac{2\pi}{q}, \frac{2\pi}{q}\cdot 2, \dots \frac{2\pi}{q}(q-1)\}$ (note $\exp(0)=1=\exp(2\pi)$)

Thus card $L^{-1}(\omega)=q$ if $q\neq 0$. (If $q=0$, L is constant so $\deg L=0$.)

so $\deg_2 f = \deg_2 L = q \pmod 2$ in any case.

Remark: see problem $\boxed{28}$ for the oriented degree.

$\boxed{17}$ $f: S^k\longrightarrow\mathbb{R}^{k+1}\setminus\{0\}$, $\hat f = \frac{f}{\|f\|}: S^k\longrightarrow S^k$.

Claim: a is a reg. value for $\hat f$ iff $f \pitchfork r$ (where $r=\{ta;\ t>0\}\subset\mathbb{R}^{k+1}\setminus\{0\}$)

Note $\hat f^{-1}(a) = \{x\in S^k;\ \frac{f(x)}{\|f(x)\|}=a\} = \{x\in S^k;\ f(x)=ta \text{ for some } t>0\} = f^{-1}(r)$.

Call this set $L_a\subset S^k$. Note $\hat f = p\circ f$ where $p:\mathbb{R}^{k+1}\setminus\{0\}\to S^k$, $p(y)=\frac{y}{\|y\|}$.

a is a reg. value for $\hat f \iff \forall x\in L_a$ $d\hat f(x): T_xS^k\to T_{\hat f(x)}S^k$ has rank k (1)

$f\pitchfork r \iff \forall x\in L_a$ $df(x)[T_xS^k]+\ell = \mathbb{R}^{k+1}$ (2)

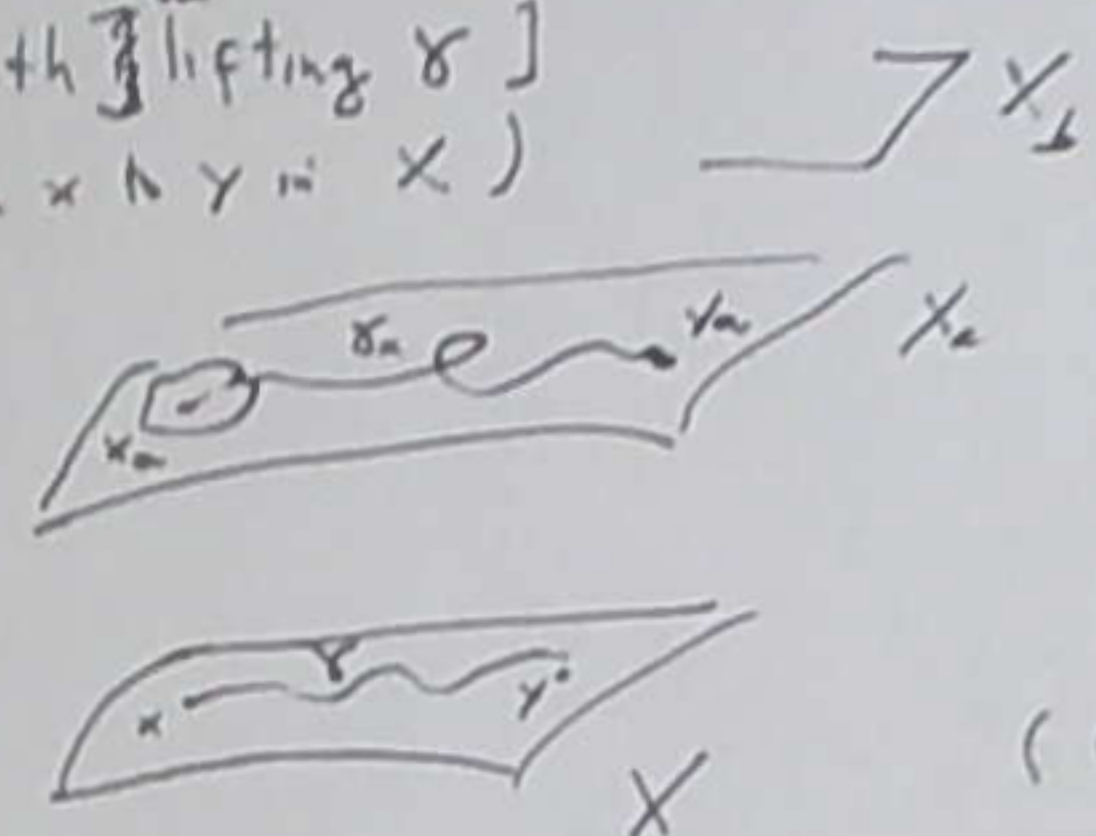
Note $d\hat f(x) = dp(f(x))\circ df(x)$ and $\underbrace{dp(y)}_{=\pi_a}:\mathbb{R}^{k+1}\to \ell^\perp = T_aS^k$ (if $y\in\ell$) is orth. proj.

Thus $d\hat f(x) = \pi_a\circ df(x)$

(1) $\iff$ $\pi_a|_{df(x)[T_xS^k]}$ is iso and $rk[df(x)]=k$ $\iff$ $\dim df(x)[T_xS^k]=k$ and $\ell\cap df(x)[T_xS^k]=0$ (since $\ell=\ker(\pi_a)$)

$\iff$ (2) QED

$\boxed{21}$ Let X be simply-connected, $\tilde{X}$ the oriented double cover.
Then $\tilde{X}$ is not connected (since X has no connected covering spaces), and consists of 2 connected components, both orientable (and diffeo to X); call them $\tilde{X} = X_a \sqcup X_b$. Given $x \in X$, fix a basis of $T_x X$, call it 'positive'.
This defines (by lifting) a positive basis $B_a(x_a)$ of $T_{x_a} X_a$ ($p(x_a) = x$).
Given $y \in X$, pick $y_a \in X_a$ with $p(y_a) = y$ [and join x_a to y_a by a path γ_a lifting γ]
(γ from x to y in X)

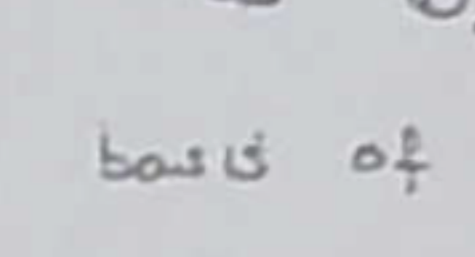


Since X_a is oriented, $\exists$ a basis of $T_{y_a} X_a$ in the same orientation class as $B_a(x_a)$. Project back by $dp(x_a)$ to a basis of $T_y X$, declared to be positive.
(Covering γ and γ_a by overlapping open sets, we see this defines an orientation of X consistently).

Thus X is orientable.

$\boxed{23}$ X: oriented, $f: X \to \mathbb{R}$ smooth, $0 \in \mathbb{R}$ reg. value.
$f^{-1}(0) = M \subset X$ codimension one submanifold. For each $p \in M$, let $v \in T_p X$ satisfy $df(p)[v] = 1$. (Note $T_p M = \ker df(p)$). Then declare $\{v_1, \dots, v_{n-1}\} \subset T_p M$ (basis) to be positive iff $\{v_1, \dots, v_{n-1}, v\}$ is a pos. basis of $T_p X$.

$\boxed{26}$ (ii) Suppose $f(t, \omega)$ is the homotopy, with $\omega \in \partial B$, $t \in [0,1]$
$f(0,\omega) = p \in Y$, $f(1,\omega) = f$
Say $B = B(0,R)$.
Let $\varphi: [0,1] \to [0,1]$ smooth,



, $\varphi \equiv 0$ in $[0, 1/4]$, $\varphi \equiv 1$ in $[3/4, 1]$.

Let $h(x) = f\left(\varphi(\|x\|/R), \frac{Rx}{\|x\|}\right)$ Then h extends $f|_{\partial B_R}$ smoothly to $\mathbb{R}^k$
(note $h(x) \equiv p$ for $\|x\| \leq R/4$)

$\boxed{27}$ Let $N_\varepsilon \subset \mathbb{R}^N$ be a tubular neighborhood of W.
Let $\varphi: \mathbb{R}_+ \to [0,1]$ be smooth with $\varphi(d) \equiv 1$ for $0 \leq d \leq \varepsilon/4$, $\varphi(d) \equiv 0$ for $d \geq \frac{3\varepsilon}{4}$
Then define $F: \mathbb{R}^N \to \mathbb{R}^{k+1}$ via

$$F(x) = \begin{cases} \varphi(d(x, W))\, f(p(x)), & x \in N_\varepsilon, \quad p: N_\varepsilon \to X \text{ closest pt map} \\ 0, & x \in \mathbb{R}^{k+1} \setminus N_\varepsilon \end{cases}$$

F is smooth and $F(x) = f(x)$ if $x \in W$.