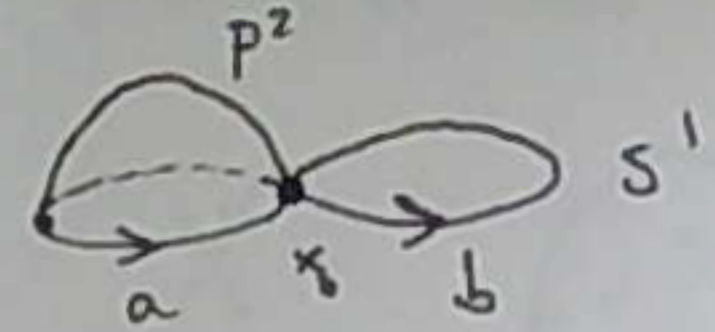
LIST 9 : solutions

$\boxed{5}$ Diagram for $P^2 \vee S^1 = X$

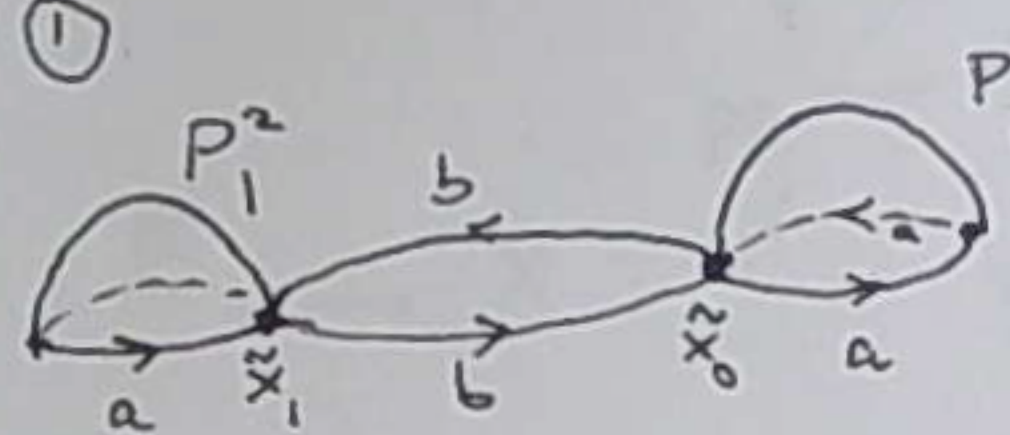


(P^2: a hemisphere w/ antipodal pts on the equator identified)

$$G = \pi_1(X, x_0) = \langle a, b \mid a^2 = 1 \rangle$$

A connected double cover corresponds to a homomorphism $\phi : G \to \mathbb{Z}_2$ (ont.), with $\Gamma = \ker \phi \triangleleft G$ the group of the cover. (Any subgroup of index 2 is normal, hence any double cover is regular.)

① $\tilde{X}_1$

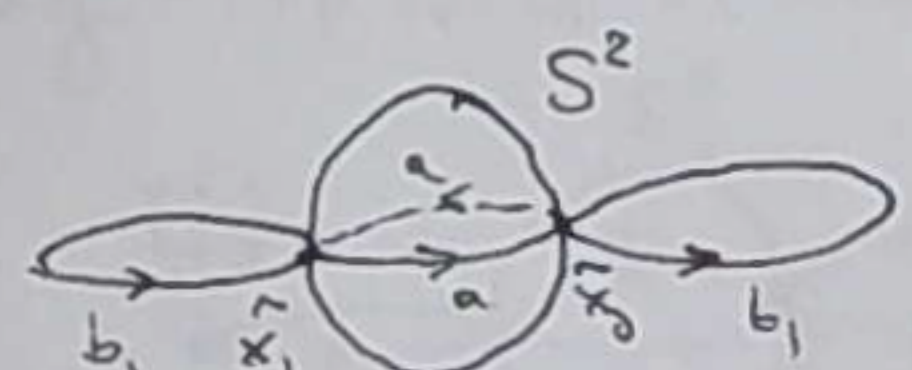


Description of $p_1 : \tilde{X}_1 \to X$

- P^2_1, P^2_2 map to P^2 : identity
- open arcs b each map to S^1 (parametrized as shown)
- $\tilde{x}_0, \tilde{x}_1$ map to x_0.

$\Gamma_1 = p_{1*}\pi_1(\tilde{X}_1, \tilde{x}_0) = \langle a, bab, b^2 \mid a^2 = 1 \rangle = \ker \phi_1$ where $\phi_1 : G \to \mathbb{Z}_2$
$a \mapsto 0, b \mapsto 1$

② $\tilde{X}_2$

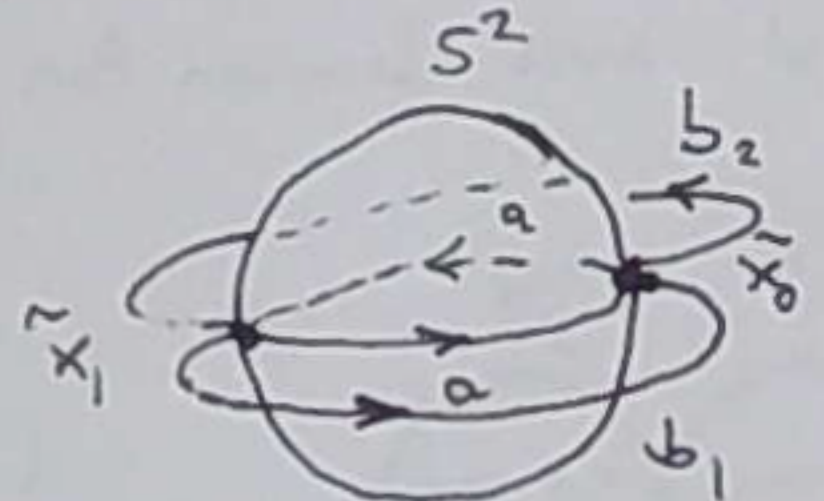


$p_2 : \tilde{X}_2 \to X$

- S^2 maps to P^2 via quotient projection
- b_1, b_2 map homeomorphically to b (as shown).
- $\tilde{x}_0, \tilde{x}_1$ map to x_0.

$\Gamma_2 = p_{2*}\pi_1(\tilde{X}_2, \tilde{x}_0) = \langle b, aba \mid a^2 = 1 \rangle = \ker \phi_2$, $\phi_2 : G \to \mathbb{Z}_2$
$a \mapsto 1, b \mapsto 0.$

③ $\tilde{X}_3$



$p_3 : \tilde{X}_3 \to X$

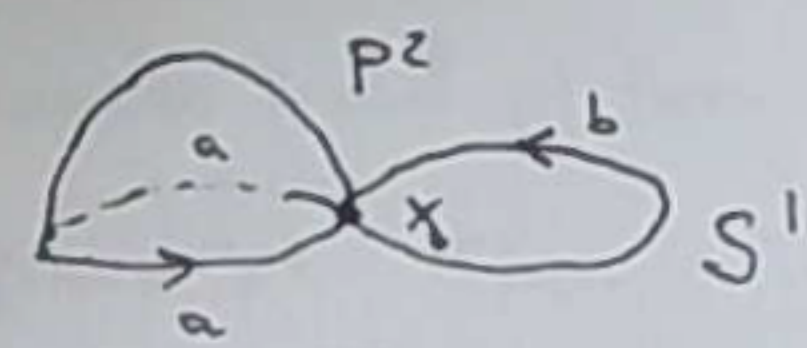
- S^2 maps to P^2 via quotient projection
- b_1, b_2 (open arcs) each map to S^1, as shown.
- $\tilde{x}_0, \tilde{x}_1$ map to x_0.

$$\Gamma_3 = p_{3*}\pi_1(\tilde{X}_3, \tilde{x}_1) = \langle b^2, ab, ba \mid a^2 = 1 \rangle = \ker \phi_3$$

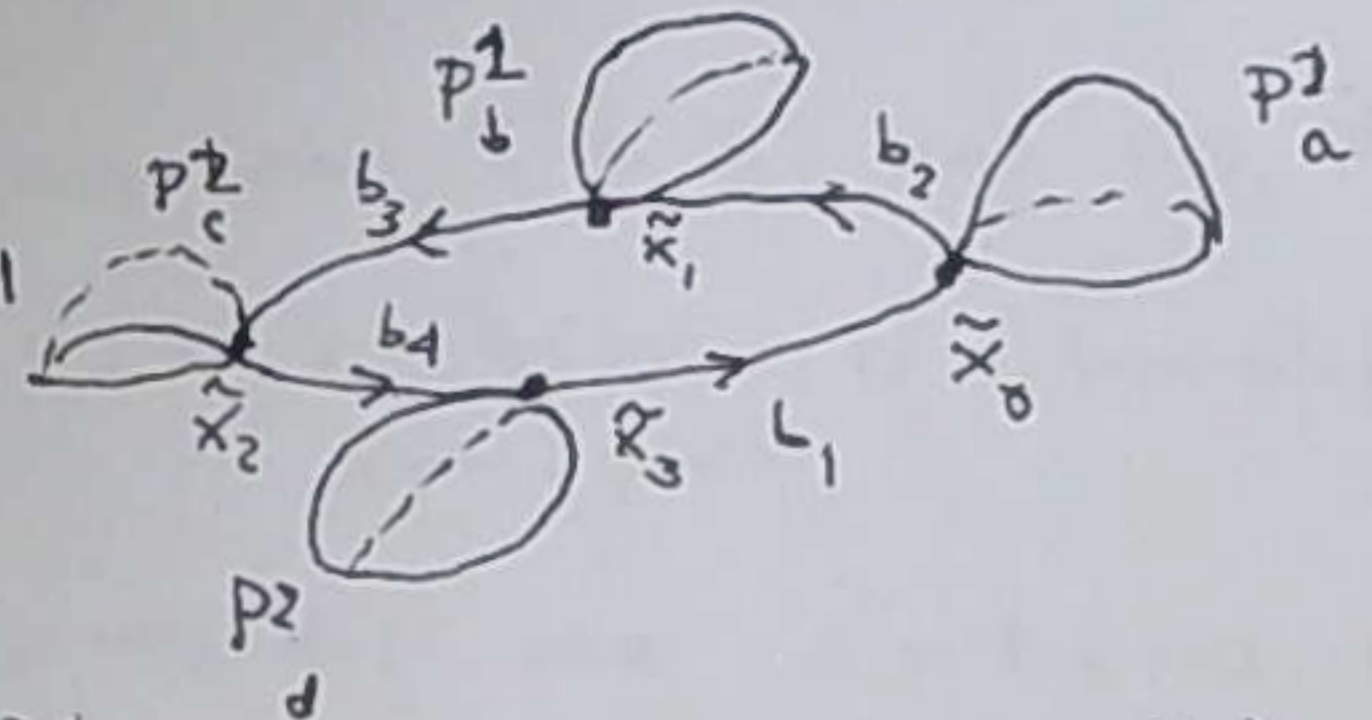
$\phi_3 : G \to \mathbb{Z}_2 \quad a \mapsto 1, b \mapsto 1$

(These are all the non-triv. homomorphisms from G to $\mathbb{Z}_2$.

[2] We use the same notation as in problem 2.

$X =$ 

$G = \pi_1(X, x_0) = \langle a, b \mid a^2 = id \rangle$

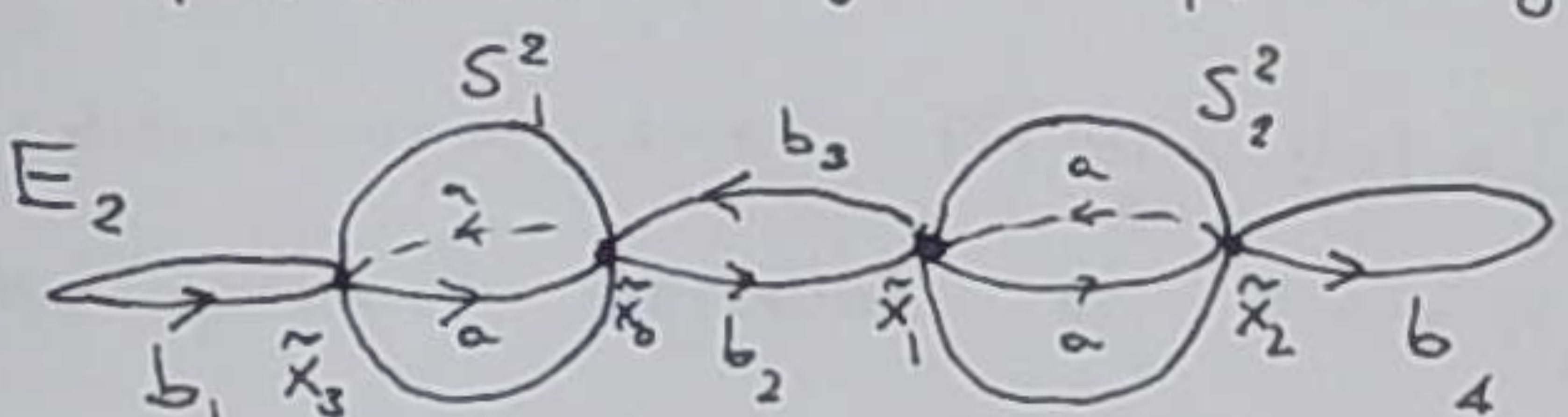
① E_1 

- each $P^2_a, \dots, P^2_d$ maps to P^2 homeomorphically
- open arcs $b_1, \dots, b_4$ each map to b as shown
- $\tilde{x}_i$ map to x_0.

$p_1 : E_1 \to X$ $\qquad \Gamma_1 = p_{1*}\pi_1(E_1, \tilde{x}_0)$

$\Gamma_1 = \langle a, bab^{-1}, b^2ab^{-2}, b^3ab^{-3}, b^4 \mid a^2 = 1 \rangle.$

Clearly $\Gamma_1 \triangleleft G$ (normal) and p_1 is regular.

② E_2 

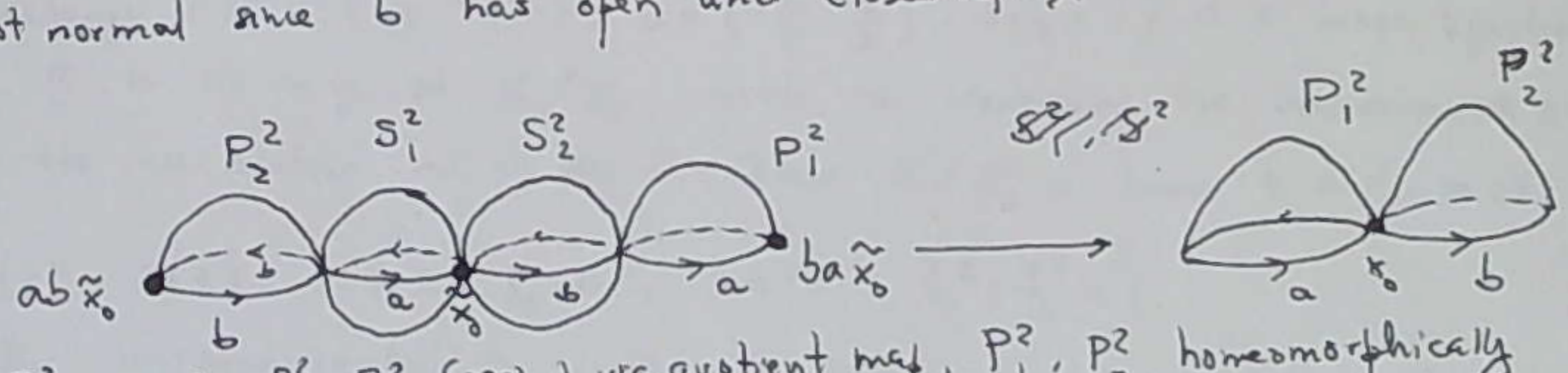
- b_1, b_4 map homeo. to S^1
- arcs b_2, b_3 map to S^1 as shown
- S^2_1, S^2_2 map to P^2 as the quotient projection (2-1)
- $\tilde{x}_i$ map to x_0

$\Gamma_2 = p_{2*}\pi_1(E_2, \tilde{x}_0)$

$= \langle b^2, aba, babab \mid a^2 = 1 \rangle$

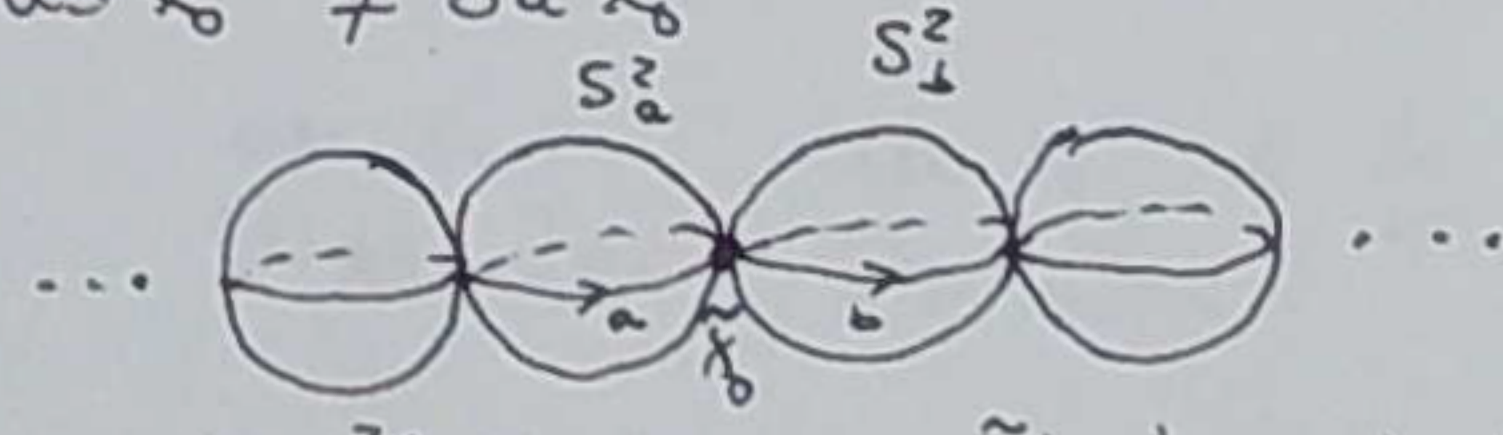
(p_2 not normal since b has open and closed lifts).

[10]

(i) 

S^2_1, S^2_2 map to P^2_1, P^2_2 (resp.) via quotient map, P^2_1, P^2_2 homeomorphically

Note $ab\tilde{x}_0 \neq ba\tilde{x}_0$.

(ii) (sketch) 

each S^2_a (resp S^2_b) maps 2-1 to P^2_a (resp P^2_b)

$n \geq 1$ $\quad (ba)^{2n}$ acts on $\tilde{X}$ by shifting to the right "by 2 spheres".

$\boxed{20}$ $B \subset \mathbb{R}^2$ closed unit disk $\partial B = S^1$ $f : B \to \mathbb{R}^2$ cont.

If $f(S^1) \subset B$, f has a fixed point in B.

Proof (Note the Broower fixed pt thm can't be used, since we don't have $f(B) \subset B$)

Assume f has no fixed pt. Then $\phi(x) = \dfrac{x - f(x)}{\|x - f(x)\|}$

is a cont. map from S^1 to S^1 and $\deg_2 \phi = 0$ since ϕ extends to B.

But $\phi_t(x) = \dfrac{x - t f(x)}{\|x - t f(x)\|}$ is a homotopy from id_{S^1} (w/ $\deg_2 = 1$) and ϕ.

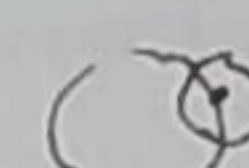
Contradiction

Rk. ϕ_t is well-def. since $x = t f(x)$ with $x \in S^1$ and $0 < t < 1$

implies $f(x) = \dfrac{x}{t}$, so $\|f(x)\| > 1$, contradicting $f(S^1) \subset B$.

$\boxed{25}$ (ii) Consider the $\mathbb{Z}_2$ action on $X = S^1 \times [-1,1]$

$\phi(\omega, t) = (-\omega, -t)$ (so $\phi^2 = id_X$)



Properly discontinuous since $\phi(B_{S^1}(\omega_0, 1/10) \times [-1,1]) \cap (B_{S^1}(\omega_0, 1/10) \times [-1,1]) = \emptyset$.

Thus the quotient projection $\pi : X \to X/\mathbb{Z}_2$ is a double cover.

consider the rectangle $R = [-\pi/2, \pi/2] \times [-1,1]$ w/ the eq. rel'n

$(-\pi/2, t) \sim (\pi/2, -t)$. The quotient space is homeo. to the Möbius strip M.

Writing X as $\{(e^{i\theta}, t) ; \theta \in [-\frac{\pi}{2}, \frac{3\pi}{2}), t \in [-1,1]\}$ we see that the

open rectangle $\mathring{R} = \{(e^{-i\theta}, t) ; \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}), t \in [-1,1]\} \subset X$ maps bijectively

under π to its image on $X/\mathbb{Z}_2$, with π identifying the boundaries $\theta = \pm\pi/2$

as in the eq. relation $\sim$ on R. Thus $X/\mathbb{Z}_2$ is homeo. to $R/\sim = M$.

$\boxed{28}$ Let $f(t) = (f_1(t), f_2(t))$, $t \in C$, $f_1^2 + f_2^2 = 1$.

By uniform continuity, we may find $t_0 < t_1 < \cdots < t_N \in C$

(with $t_0 = \inf C$, $t_N = \sup C$) with the oscillation bounds:

$$\operatorname*{osc}_{[t_i, t_{i+1}] \cap C} f_1 \,,\ \operatorname*{osc}_{[t_i, t_{i+1}] \cap C} f_2 \le \frac{1}{10}.$$

28 cont

By Tietze extension, we may extend f_1, f_2 to each interval $[t_i, t_{i+1}]$ with the same oscillation bounds, obtaining $g_1, g_2 : [t_0, t_N] \to \mathbb{R}$ (cont.) satisfying:

$$\underset{[t_i, t_{i+1}]}{\operatorname{osc}} g_1 , \quad \underset{[t_i, t_{i+1}]}{\operatorname{osc}} g_2 \leq \frac{1}{10} \qquad (i = 0, \dots, N-1).$$

In particular: $g_1^2 + g_2^2 \neq 0$ on $[t_0, t_N]$ (since $f_1^2 + f_2^2 = 1$ on C and $f_j(t_i) = g_j(t_i)$ $j = 1, 2$)

Normalizing, we obtain:

$$\hat{g} : [t_0, t_N] \to S^1 \text{ (cont.) extending } f.$$

By the lifting theorem, $\hat{g}$ lifts to $g : [t_0, t_N] \to \mathbb{R}$ (cont.)

So $\hat{g}(t) = e^{ig(t)}$. In part., for $t \in C$: $e^{ig(t)} = f(t)$.

34 (i) critical points: $p'(z) = 0 \to z \in \{4, -1\} = C$

Let $F_2 = p(C)$, $F_1 = p^{-1}(p(C))$ $(\# F_2 = 2, \# F_1 = 4)$

Then $p : \mathbb{C} \setminus F_1 \to \mathbb{C} \setminus F_2$ is a local diff., and surjective (Fund. Thm Algebra).

Also p is proper: if $z_n \to \infty$, $|p(z_n)| \geq |z_n|^3$ for n suff. large, so $p(z_n) \to \infty$.

(ii) is similar.

37 The hypothesis implies

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "p_1"'] & X \arrow[d, "p_2"] \\ & X/G \end{tikzcd}$$

commutes (p_i are the quotient projections).

We need to check $f_* : \pi_1(X, x_0) \to \pi_1(X, f(x_0))$ is onto

Let $\alpha \in \pi_1(X, f(x_0))$. Let $\beta = p_{2*}(\alpha) \in \pi_1(X/G, y)$ $y = p_2 \circ f(x_0) = p_1(x_0)$.

Let $\gamma \in \pi_1(X, x_0)$ be the unique element s.t. $p_{1*}(\gamma) = \beta$ (p_{i*} are injective). Then $p_{2*}(f_* \gamma) = \beta$, so $f_* \gamma = \alpha$.