TOPOLOGY PRELIM REVIEW 2021: LIST ONE

Topic: Countability properties.

1. (i) Let $E \subset X$, E' be the set of cluster points of E. Then $\bar{E} = E \cup E'$.

(ii) If X is first countable and $a \in E'$, then one may find a sequence $(x_n)_{n\geq 1}$ in E, so that $\lim x_n = a$.

2. (*Sorgenfrey line*, denoted $\mathbb{R}_l$) (i) The collection of subsets of $\mathbb{R}$ $\mathcal{B} = \{[a,b); a < b\}$ (left-closed intervals) define the basis of a topology on $\mathbb{R}$.
(ii) This topology is Hausdorff, and is finer than the usual topology on $\mathbb{R}$.

3. In a metrizable space X, $E \subset X$ is closed iff E is sequentially closed.

4. (i) (X, d) and $(X, \min\{d, 1\})$ are equivalent metric spaces (i.e., the identity map is a homeomorphism.)
(ii) $d_1(x,y) = |x-y|$ and $d_2(x,y) = |x^3 - y^3|$ define equivalent metrics oon $\mathbb{R}$ which are not quasi-isometric.

5. (i)X (topological space) second countable $\Rightarrow$ X first countable and separable.
(ii) X metrizable and separable $\Rightarrow$ X second-countable.

6*. X (topological space) second countable $\Rightarrow$ any open cover admits a countable subcover (Lindelöf)

7*. The Sorgenfrey line is first countable and separable, but not second countable (and hence is not metrizable.)

8*. Let $X = \mathcal{F}(\mathbb{R}, \mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}\}$, the space of *all* functions from $\mathbb{R}$ to $\mathbb{R}$, with the product topology (as $\mathbb{R}^{\mathbb{R}}$.)
(i) Describe explicitly a basis for this topology (and verify that it satisfies the conditions for a basis);
(ii) Show that $\lim f_n = f$ in this topology iff $f_n(t) \to f(t)$, $\forall t \in \mathbb{R}$ (pointwise).

9*. (i) Show that $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is not first countable (hence not metrizable).

(ii) Let $E \subset \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the set of characteristic functions of finite sets. The constant function 1 is in E', but there is no sequence $f_n \in \mathcal{F}$ so that $\lim f_n = f$.

Remark. The same considerations apply to $\mathcal{F}_p(\mathbb{R}, [0,1]) = [0,1]^{\mathbb{R}}$ or $\mathcal{F}_p([0,1]; [0,1])$ (with pointwise convergence, or equivalently the product topology). This gives examples of spaces that are compact (by Tychonoff's theorem), but not metrizable.

10. (i) A separable metric space cannot contain an uncountable discrete set.
(ii) $C(\mathbb{R}; [0,1])$ (with the uniform metric) is a metric space without a countable basis (equivalently, not separable.) *Hint:* For $S \subset \mathbb{Z}$, define $f(n) = 1$ if $n \in S$, $f(n) = 0$ if $n \in \mathbb{Z} \setminus S$, and continuous and linear otherwise. Then the set $\{f_S; S \subset \mathbb{Z}\}$ is uncountable and discrete: $d(f_S, f_T) = 1$ if $S \neq T$.

In particular $C(\mathbb{R},[0,1])$ (w/ uniform metric) is not compact (since compact+metrizable $\Rightarrow$ separable), while $\mathcal{F}_p(\mathbb{R},[0,1]) = [0,1]^{\mathbb{R}}$ (pointwise convergence, or product topology) is compact (Tychonoff.)

11* Let X be an infinite set, (M,d) a metric space with at least two elements. Then $\mathcal{B}(X;M)$ (the space of bounded functions from X to M, with the uniform metric) does not have a countable basis.

For example, the metric space $\mathcal{F}(\mathbb{N};\{0,1\})$ (with the uniform metric) is not separable. But in fact this space is *discrete*, so a better example is $\mathcal{F}(\mathbb{Z},[0,1])$ (with the uniform metric).

12. In a second countable space, any family $\{U_\lambda\}$ of nonempty disjoint open sets is necessarily countable. (*Hint:* The subspace $S = \bigcup_\lambda U_\lambda$ is second countable, and has an obvious open cover. Use Lindelöf's theorem, and consider if this cover has any subcovers.)

13. (i) The continuous surjective image of a separable space is separable. That is, if $f : X \to Y$ is continuous and onto, and X is separable, then so is Y.
(ii*) The space of Lipschitz functions $f : [0,1] \to \mathbb{R}$, with the topology defined by the norm:

$$||f|| = |f(0)| + [f], \quad [f] = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}$$

is not separable.

An important class of separable metric spaces is given by the following.

Theorem: If K is compact metric and Y is separable metric, $X = C(K;Y)$ (with the uniform metric) is separable.

Outline of proof. For each n, fix a cover of K by finitely many closed balls K_i^n, with diameter at most $1/n$ (using the fact K is totally bounded):

$$K = K_1^n \cup \ldots \cup K_p^n, \quad p = p(n).$$

Let $\mathcal{B}$ be a countable basis for Y. Denote by σ an arbitrary p-tuple of open sets in $\mathcal{B}$, $\sigma = (B_1, \ldots, B_p)$. Given n and σ, define:

$$A(n,\sigma) = \{f \in X; f(K_i^n) \subset B_i, i = 1, \ldots, p.\}$$

It is easy to show the $A(n,\sigma)$ are a countable family of open subsets of X. To show this family defines a basis, we need to prove that given $f \in X$, $\epsilon > 0$ we may find n and σ so that:

$$f \in A(n,\sigma) \subset B(f,\epsilon).$$

The compact $f(K)$ is covered by finitely many open sets in $\mathcal{B}$, each of diameter at most $\epsilon/2$. Let $\eta > 0$ be a Lebesgue number of this cover (where we may assume $\eta < \epsilon/2$). By uniform continuity, we may pick $n \geq 1$ large enough so that

$$C \subset K \text{ compact }, diam(C) \leq 1/n \Rightarrow diam f(C) \leq \eta,$$

and hence $f(C)$ is contained in a single set of this cover. In particular, we may find $\sigma = (B_1, \ldots, B_p)$ so that $f(K_i^n) \subset B_i$ for $i = 1, \ldots, p$. That is, $f \in A(n, \sigma)$. We may also show (left to the reader) $diam(A(n, \sigma)) \leq \epsilon$, which gives the inclusion into $B(f, \epsilon)$.

For example, $C([0,1];[0,1])$ (uniform topology) is separable metric, while $\mathcal{F}([0,1];[0,1])$ (all functions, pointwise convergence) is compact (Tychonoff), but not first countable; in particular, neither second countable nor metrizable.