TOPOLOGY PRELIM REVIEW 2021: LIST THREE

Topic 1: Baire property and G_δ sets.

Definition. X is a *Baire space* if a countable intersection of open, dense subsets of X is dense in X.

Complete metric spaces and locally compact spaces are Baire spaces.

Def. X is *locally compact* if for all $x \in X$, and all open U_x, there exists V_x with compact closure $\overline{V}_x \subset U_x$.

1. X is locally compact $\Leftrightarrow$ for all $C \subset X$ compact, and all open $U \supset C$, there exists V open with compact closure, so that: $C \subset V \subset \overline{V} \subset U$.

2. *Def:* A set $E \subset X$ is *nowhere dense* in X if its closure $\overline{E}$ has empty interior.

Show: X is a Baire space $\Leftrightarrow$ any countable union of nowhere dense sets has empty interior.

3. A complete metric space without isolated points is uncountable. (*Hint:* Baire property, complements of one-point sets.)

4. *Uniform boundedness principle.* X complete metric, $\mathcal{F} \subset C(X)$ a family of continuous functions, bounded at each point:

$$(\forall a \in X)(\exists M(a) > 0)(\forall f \in \mathcal{F})|f(a)| \leq M(a).$$

Then there exists a nonempty open set $U \subset X$ so that $\mathcal{F}$ is eqüibounded over U–there exists a constant $C > 0$ so that:

$$(\forall f \in \mathcal{F})(\forall x \in U)|f(x)| \leq C.$$

Hint: For $n \geq 1$, consider $A_n = \{x \in X; |f(x)| \leq n, \forall f \in \mathcal{F}\}$. Use Baire's property.

An important application of 4. is the *uniform boundedness principle* for families of bounded linear operators $\mathcal{F} \subset \mathcal{L}(E;F)$, where E, F are Banach spaces.

5. If $f : X \to Y$ is any function (X topological space, Y metric space), the set of continuity $C_f \subset X$ is a G_δ set.

Hint: Let U_n be the union of all open sets V such that the diameter of $f((V)$ is less than $1/n$. Show $C_f = \bigcap_{n\geq 1} U_n$.

6. (i) $\mathbb{Q}$ is not a G_δ subset of $\mathbb{R}$ (hence there are no functions $f : \mathbb{R} \to \mathbb{R}$ with continuity set $C_f = \mathbb{Q}$).

(ii) The irrationals $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ are a G_δ set.

(iii) In a separable Baire space without isolated points, no countable dense subset is a G_δ.

Theorem. X: Baire space, Y: metric space, $f_n : X \to Y$ continuous, $f_n \to f$ pointwise. Then C_f is a dense G_δ set in X.([Munkres, p. 297]).

7. Let $f_n : \mathbb{R} \to \mathbb{R}$ continuous. Suppose $f_n \to f$ pointwise on $\mathbb{R}$. Then f is continuous at uncountably many points of $\mathbb{R}$.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Show that the derivative $g = f' : \mathbb{R} \to \mathbb{R}$ is the pointwise limit of a sequence of continuous functions. (As a consequence, the set of continuity of g is a dense G_δ set.)

Remark: The converse is true: for any dense G_δ set $A \subset \mathbb{R}$, there exists $f : \mathbb{R} \to \mathbb{R}$ differentiable, so that the continuity set of f' is A.(*BAMS*, 4/2019.)

9* Any open subset A of a complete metric space M is homeomorphic to a complete metric space.

Outline: (i) Let $f : M \to \mathbb{R}$ a continuous function vanishing exactly on $M \setminus A$, say $f(x) = d(x, M \setminus A)$. Then $g : A \to \mathbb{R}, g(x) = 1/f(x)$ is continuous on A. Then $G = \{(x,t); x \in A, t = g(x)\}$ is closed in $M \times \mathbb{R}$, since $G = \{(x,t); x \in M, tf(x) = 1\}$. Thus G is a complete metric space. But the projection onto A is a homeomorphism from G to A. Explictly, the metric on A:

$$d_1(x,y) = |\frac{1}{d(x, M \setminus A)} - \frac{1}{d(y, M \setminus A)}| + d(x,y)$$

is complete, and equivalent to the original metric on A (defines the same topology.)

Topic 2: Proper (or Heine-Borel) metric spaces.

Def. (X,d) is *proper* (or HB) if closed bounded sets are compact.

Ex.1 $\mathbb{R}^n$ is HB. (Since bounded sets are totally bounded.)

Ex.2 For X compact metric, $C(X)$ (with the sup norm) is complete separable metric, but not HB. (Prove this.)

10*. Let (X,d) be a proper metric space. Then X is complete, locally compact and σ-compact.

The converse is not true: it is easy to destroy the HB property.

Ex. 3. Let (X,d) be a non-compact metric space. Then the metric $d_{min} = \min\{d, 1\}$ is equivalent to d (same topology), locally coincides with d (in a neighborhood of the diagonal in $X \times X$), but is not HB. In particular, (X, d_{min}) is locally compact, σ-compact and complete, if d is.

11. A metric space (X,d) is proper if and only if the distance function to a point $x \mapsto d(x, x_0)$ is a proper function on X (preimage of compact is compact.) Hence the name.

Theorem [Williamson-Janos]. If (X,d) is locally compact, σ-compact, there exists an equivalent metric (same topology) which is HB.

Idea of proof. Let (K_n) be a compact exhaustion of X, $K_n \subset int(K_{n+1})$. Let $f_n : X \to [0,1]$ continuous satisfy: $f_n \equiv 0$ on K_n, $f_n \equiv 1$ on K_{n+1}^c. Consider the metric on X:

$$d'(x, y) = d(x, y) + \sum_{n \geq 1} |f_n(x) - f_n(y)|.$$

12. Show that d and d' are equivalent on X. (It suffices to show that $\lim x_n = x$ with respect to d iff this holds with respect to d'.)

Note that any d'-bounded set must be contained in some K_n, so (X, d') is HB. In particular complete, even if (X, d) isn't. This gives a proof that locally compact, σ-compact metric spaces are completely metrizable. (This also follows from the Alexandroff compactification.)