TOPOLOGY PRELIM REVIEW 2021: LIST FOUR

Topic 1: Countable compactness and sequential compactness.

Definitions. X is *countably compact* if any countable open cover admits a finite subcover.

Thus any compact space is countably compact, and on a Lindelöf space the concepts are equivalent (for example on any 2nd countable space.)

X is *sequentially compact* if any sequence on X admits a convergent subsequence.

A point $z \in X$ is an *accumulation point* of a sequence (x_n) in X if any neighborhood of z contains infinitely many points of the sequence.

1. (i) X is compact iff any family $\{C_\lambda\}_{\lambda\in A}$ of closed subsets of X with the *finite intersection property* has non-empty intersection:

$$\Big(\bigcap_{\lambda\in F} C_\lambda \neq \emptyset \quad \forall F\subset A \text{ finite }\Big) \Rightarrow \bigcap_{\lambda\in A} C_\lambda \neq \emptyset.$$

(ii) $K_1 \supset K_2 \supset \ldots$, K_n compact Hausdorff and nonempty $\Rightarrow \bigcap_{n\geq 1} K_n \neq \emptyset$.

2. If X is first-countable, $z \in X$ is an accumulation point of (x_n) iff some subsequence of (x_n) converges to z.

3. If X is countably compact, any sequence in X has an accumulation point. As a consequence, if X is countably compact (in particular, if X is compact) and first-countable, X is sequentially compact.

4. If X is sequentially compact, X is countably compact. (Hence for first-countable spaces, these concepts are equivalent.)

5. The space X of all functions from $[0,1]$ to itself (with the topology of pointwise convergence) is compact (by Tychonoff's theorem), but not first-countable. And indeed it is not sequentially compact: exhibit a sequence $f_n \in X$ with no convergent subsequence.

6. If X is second countable and sequentially compact, X is compact.

Summary: 1st countable + compact $\Rightarrow$ sequentially compact;
2nd countable + sequentially compact $\Rightarrow$ compact.

7. (i) Compact Hausdorff spaces are normal.
(ii) Second countable, regular (includes Hausdorff) spaces are normal.
Remark: It follows from Urysohn metrization that compact, second countable Hausdorff spaces are metrizable.

Topic 2: Compactness in metric spaces.

8. (X,d) compact metric is sequentially compact; in particular, X is complete.

Def. (X,d) is totally bounded: for each $R>0$, finitely many balls of radius R cover X. (Thus: compact metric spaces are totally bounded.)

9. (i) In a totally bounded metric space, any sequence has a Cauchy subsequence (nested balls argument.)

(ii) If (X, d) is totally bounded and complete, X is sequentially compact.

10. A totally bounded metric space is separable (hence second-countable.)

11. If (X, d) is a sequentially compact metric space, X is complete and totally bounded.

So far we see that a compact metric space is sequentially compact; and that a metric space is sequentially compact iff it is complete and totally bounded. To close the circle, we need:

Lebesgue number lemma. Any open cover $\mathcal{F}$ of a sequentially compact metric space has a *Lebesgue number*: $L > 0$ so that any subset $C \subset X$ with diameter less than or equal to L is contained in a set U of the cover.

Proof. If false, there exists a cover $\mathcal{C} = \{U_\lambda\}$ and for each $n \geq 1$ a set $S_n \subset X$ with diameter at most $1/n$, but not contained in any U_λ. Let $x_n \in S_n$; passing to a subseq. (if needed) we have $x_n \to x$, and may choose λ_0 and $\epsilon > 0$ so that $B(x, \epsilon) \subset U_{\lambda_0}$. Let $N \geq 1$ be large enough that, for $n \geq N$, $\frac{1}{n} < \frac{\epsilon}{2}$ and $d(x_n, x) < \frac{\epsilon}{2}$. Then if $y \in S_n$:

$$d(y, x) \leq d(y, x_n) + d(x_n, x) < \frac{1}{n} + \frac{\epsilon}{2} < \epsilon,$$

so $S_n \subset B(x, \epsilon) \subset U_{\lambda_0}$, contradiction.

12. Any sequentially compact metric space (X, d) is compact.

CONCLUSION: For a metric space, 'compact', 'sequentially compact', 'countably compact' and 'complete and totally bounded' are all equivalent. Compact metric spaces are separable and second countable; and any open cover admits a Lebesgue number.

13*. A metric space X is separable if and only if it is homeomorphic to a subset of a compact metric space. (*Hint:* Embedding in the Hilbert cube.)

Topic 3: Locally compact spaces.

Recall from review list two:

Def. X is *locally compact* if for all $x \in X$, and all open U_x, there exists V_x open with compact closure $\overline{V}_x \subset U_x$. (In particular, locally compact Hausdorff spaces are regular.)

(Problem 1 on list two). X is locally compact $\Leftrightarrow$ for all $C \subset X$ compact, and all open $U \supset C$, there exists V open with compact closure, so that: $C \subset V \subset \overline{V} \subset U$.

Def. A metric space (X, d) is *locally separable* if for each $x \in X$ there exists an open ball $B(x, r_x)$ containing a countable dense subset.

Proposition 1. If (X, d) is connected and locally separable, then X is separable. (Proved in M561.)

14. If (X, d) (metric) is connected and locally compact, then X is separable (and second countable.)

A natural question is: which subsets of locally compact Hausdorff spaces are locally compact? The following propositions were proved in M561:

Def. $S \subset X$ is *locally closed* if any $x \in S$ has a neighborhood $U_x \subset X$ (open), so that $U_x \cap S$ is *closed* in U_x (that is, $U_x \cap S = U_x \cap F_x$, for some F_x closed in X).

Proposition 2. S is loc. closed $\Leftrightarrow S = U \cap F$, where U is open in X, F is closed in X. (So this could be taken as the definition of 'locally closed'.)

Proposition 3. X loc. compact Hausdorff, $S \subset X$ loc. closed $\Rightarrow S$ is loc. compact. And conversely: locally compact subsets of X are locally closed.

Cor. Closed sets, as well as open sets, of locally compact Hausdorff spaces are locally compact (for the induced topology).

15.* (i) X loc. compact Hausdorff, $S \subset X$ dense in X and locally compact $\Rightarrow S$ is *open* in X. (In particular, any locally compact metric space is open in its completion.)

(ii) Any locally compact metric space is homeomorphic to a complete metric space (that is, its topology is given by an equivalent, complete metric.)

Topic 4: Alexandrov compactification/proper maps and perfect maps

A locally compact Hausdorff space X admits an *Alexandrov compactification* X^*, meaning a compact Hausdorff space X^* and an embedding $\varphi : X \to X^*$ with $X^* \setminus \varphi(X) = \{\omega\}$, the 'point at infinity'. Neighborhoods of ω in X^* have the form $\{\omega\} \sqcup U$ (disjoint union), where $U \subset X$ is the complement of a compact subset of X. [Proof given in lecture.]

Such a compactification is unique: if $\psi : X \to \tilde{X}$ is a second one, $\tilde{X} = X \sqcup \{\tilde{\omega}\}$, one may find a homeomorphism $h : X^* \to \tilde{X}$, $h(\omega) = \tilde{\omega}$, so that $\varphi = \psi \circ h_{|X}$.

16. Use the existence of the Alexandrov compactification to prove that locally compact Hausdorff spaces are completely regular.

The Alexandrov compactification of a loc. cpt Hausdorff space X is *countable at infinity* if the point at infinity has a countable basis of neighborhoods.

17. This happens iff X is σ*-compact*: $X = \bigcup_{i \geq 1} K_i$, $K_i \subset X$ compact, which may be assumed to be increasing, $K_i \subset int(K_{i+1})$.

18*. Let X be locally compact metric. The following are equivalent:
(i) X has a countable basis (in particular, Lindelöf.);
(ii) X is σ-compact;
(iii) X^* is metrizable.

Def. Let X, Y be both locally compact Hausdorff. A continuous map $f : X \to Y$ is *proper* if the preimage of any compact set is compact.

19. f extends to a continuous map $F : X^* \to Y^*$ of the Alexandrov compactifications (with $F(\omega_X) = \omega_Y$) iff f is proper.

Remark on problem 19. A related concept is that of *perfect map* [Munkres, p.199]: A continuous surjective map $f : X \to Y$ is *perfect* if it is closed and all level sets (or 'fibers') $f^{-1}(y)$ are compact.

This implies the properties (i) Hausdorff; (ii) regular; (iii) locally compact; (iv) second countable are inherited by Y, if satisfied by X.

20. Prove (i) and (ii) of this claim.

21. Let $f : X \to Y$ be continuous, surjective and closed.

(i) For each $y \in Y$ and any $U \subset X$ open neighborhood of the preimage (level set) $f^{-1}(y)$, there exists $V \subset Y$ open neighborhood of y so that $f^{-1}(V) \subset U$. (In fact this 'continuity of level sets' characterizes closed maps.)

(ii) Let $y \in Y$, let $U \subset X$ be an open neighborhood of $f^{-1}(y)$. Then $f(U)$ contains an open neighborhood $V \subset Y$ of y.

22. Perfect maps are proper. *Hint:* Problem 21(i).

Remark. The converse is true, under the hypothesis the topologies of X and Y are *compactly generated* ([Munkres p. 283]): a subset $A \subset X$ is open in X if $A \cap C$ is open in C, for each $C \subset X$ compact subspace. As proved in [Munkres, p.283]: locally compact spaces and first countable spaces (in particular, metric spaces) are compactly generated.

5. Supplementary problems.

23. (i) If $Y_1, Y_2, \ldots$ are sequentially compact, then $Y = \Pi_{n\geq 1} Y_n$ is sequentially compact.

(ii) If N is countable and Y is sequentially compact, $\mathcal{F}_p(N, Y)$ (pointwise convergence) is sequentially compact.

24. (Application of 23–'Helly's theorem'). Let $X \subset \mathbb{R}$ be arbitrary, $f_n : X \to [a, b]$ a sequence of monotone functions (say nondecreasing.) Then (f_n) has a convergent subsequence (pointwise in X). *Hint:* Show f_n has a subsequence converging pointwise in a countable dense subset of X, then use monotonicity.

25. Let Y be compact Hausdorff, X arbitrary. Then $\pi : X \times Y \to X$ is a closed map. (*Hint:* tube lemma). This is false if Y is not compact.

26. Let $f : X \to Y$ be a map (X a space, Y compact Hausdorff). (i) If the graph $\Gamma_f \subset X \times Y$ is closed, f is continuous. (Hint: if $V \subset Y$ is a nbd of $f(x_0)$, $C = \Gamma_f \cap (X \times V^c)$ is closed; consider its image under π, the standard projection from $X \times Y$ to X.) This is false if Y is not compact.

(ii) If f is continuous, Γ_f is closed (Y compact not needed, just Hausdorff.)

27. Prove the *weak Tychonoff theorem:* If M_i are compact metric spaces, then the product $\Pi_{i\geq 1} M_i$ is a compact metric space.

28. A continuous map $f : M \to N$ of metric spaces M, N is proper iff the image $(f(x_n))$ of a sequence (x_n) on M without convergent subsequences has the same property.

HINTS

3: If (x_n) is a sequence without any accumulation points, each x_n has a neighborhood U_n with no other points of the sequence. Let $A = \{x_1, x_2, \ldots\}$, a closed subset of X (why?). Adding A^c to the U_n gives an open covering of X, which has no finite subcover.

4: Let $\{U_n\}_{n \geq 1}$ be a countable open cover of X. If it has no finite subcover, we may define a sequence (x_n) in X taking:

$$x_1 \in X \setminus U_1, x_2 \in X \setminus (U_1 \cup U_2), \ldots, \quad x_n \in X \setminus (\bigcup_{i=1}^{n} U_i).$$

Note $x_n \notin U_i$ if $n \geq i$: each U_i has only finitely many sequence elements. But if $z \in X$, z is in some U_{n_0}.

5: Let f_n be the function (with image in $\{0,1\}$) that assigns to each $x \in [0,1]$ the nth. digit in its base 2 expansion (terminating in 0s if x is a dyadic rational). For any increasing sequence $(n_k)_{k \geq 1}$, Let $x_0 \in [0,1]$ have the binary expansion $0.a_1 a_2 . a_3 \ldots$: $a_k = 0$ if k is even, $a_k = 1$ if k is odd. Then $f_{n_k}(x_0)$ does not converge.

6. X is first countable (hence countably compact) and Lindelöf.

7. Given $A, B \subset X$ closed disjoint, use regularity and the Lindelöf property to find countable open covers $(U_i), (V_j)$ of A, B with $U_i \cap V_i = \emptyset$. Then 'correct' the U_i, V_i (by successively removing appropriate unions) so as to obtain new countable open families W_i, Z_i, containing A, B (resp.) and with disjoint unions.

8. This follows from the fact X is first countable (and countably compact). For a direct proof, if a sequence with not convergent subsequence exists, then for each $x \in X$ there is an open ball $B(x, r_x)$ including only finitely many sequence elements; taking a finite subcover leads to a contradiction. (This uses first countability too.)

11. If not, one may find $R > 0$ and $x_1 \in X, x_2 \notin B(x_1, R), \ldots x_n \quad / \; in B(x_1, R) \cup \ldots B(x_{n-1}, R)$, so $d(x_{n+1}, x_i) \geq R$ for $i = 1, \ldots, x_n$: this sequence has no convergent subsequence.

12. If not, we may find a sequence of closed sets C_n with diameter less than $1/n$, not contained in any open set in $\mathcal{F}$, and $x_n \in C_n$. Then a subsequence $x_{n_i} \to z \in U$, $U \in \mathcal{F}$ open. But also $B(z, \epsilon) \subset U$ for some $\epsilon > 0$, and for i large, $C_{n_i} \subset U$ (show this). Contradiction.

12. Use the fact X is totally bounded: given an arbitrary open cover, consider its Lebesgue number L, and cover X by finitely many balls of radius $L/3$.

15. S is loc. closed in X. So $S = F \cap U$ with F closed in X, U open in X. But then S is dense in U (since U is open in X) and closed in U; so $S = U$.