TOPOLOGY PRELIM REVIEW 2021: LIST FIVE (version date: 6/29/2021)

Topic: topologies in spaces of maps, Arzela-Ascoli, Stone-Weierstrass.

1*. X locally compact, σ-compact; (Y, d) metric. Then the u.o.c. topology on $\mathcal{F}(X, Y)$ is metrizable, with a complete metric if d is complete.

Hint: Let (K_i) be a compact exhaustion of X, $K_i \subset int(K_{i+1})$. Letting $\mathcal{F}_i = \mathcal{F}(K_i, Y)$ with the uniform topology, show $\mathcal{F}(X, Y)$ (u.o.c.) is homeomorphic to a closed subset of the product of the $\mathcal{F}_i$.

2. The compact-open topology in $C(X, Y)$ is Hausdorff if Y is Hausdorff, regular if Y is regular.

3. X is locally compact, σ-compact. $f_n : X \to \mathbb{R}^k$. If (f_n) is equicontinuous on compact sets and bounded at each point, there exists a subsequence converging u.o.c. to a continuous function. (You may use the A-A theorem.)

Stone's theorem (1948). $\mathcal{A} \subset C(X)$: algebra separating points, not all vanishing at any point $\Rightarrow$ $\mathcal{A}$ is dense in $C(X)$ (for u.o.c convergence; X loc. compact and σ-compact.)

4. Let $X = [0, \infty)$. Show that for each continuous $f : X \to \mathbb{R}$ there exists a sequence of the form:

$$p_k(x) = \sum_{n=0}^{n_k} a_n e^{-nx}$$

such that $p_k \to f$ uniformly on compact sets.

5. If $f \in C(\mathbb{R}^n)$, there exists a sequence p_j of polynomials in n variables so that $p_j \to f$ u.o.c. in $\mathbb{R}^n$. If $f(0) = 0$, we may require the approximating polynomials to satisfy $p_j(0) = 0$ for all j.

Stone-Weierstrass for lattices. Let $\mathcal{L} \subset C(X)$be a lattice (closed under min and max of finite subsets) with the two-point interpolation property (given $p \neq q$ in X, $a, b \in \mathbb{R}$, $\exists f \in \mathcal{L}$, $f(p) = a, f(q) = b$). Then $\mathcal{L}$ is dense in $C(X)$ (for u.o.c. convergence; X loc. compact and σ-compact.)

6. The set of continuous, piecewise linear functions is dense in $C(\mathbb{R})$ (for the u.o.c. topology). *Hint:* use the version of Stone-Weierstrass for lattices of continuous functions.

7. Let X be locally compact Hausdorff. If X is σ-compact with compact exhaustion $(K_n)_{n\geq 1}$, define a metric on $C(X)$ by:

$$\rho(f, g) = \sum_{n=1}^{\infty} \rho_n(f, g),$$

$$\rho_n(f, g) = \min\{\frac{1}{2^n}, \sup_{x\in K_n} |f(x) - g(x)|\}.$$

Show that ρ metrizes the u.o.c. topology on $C(X)$.

8. X: metric space, E: Banach space. $f: X \to E$ continuous is *compact* if $A \subset X$ bounded $\Rightarrow \overline{f(A)}$ is compact. f is *finite-dimensional* if $f(X)$ is contained in a finite-dimensional subspace of E. Clearly finite dimensional, continuous bounded maps are compact.

(i) Suppose $f_n : X \to E$ is a sequence of compact maps, converging (uniformly on each bounded subset of X) to a map $f : X \to E$. Then f is a compact map.

(ii) Any compact map $f : X \to E$ is, on each bounded subset $U \subset X$, a uniform limit of finite-dimensional maps. That is, given $\epsilon > 0$ one may find $f_\epsilon : U \to Y_\epsilon$, $Y_\epsilon \subset E$ finite-dimensional, so that $||f_\epsilon(x) - f(x)|| < \epsilon$ for $x \in U$.

Hint: Since $\overline{f(U)}$ is compact, we may cover it by finitely many open balls $B_i = B(y_i, \epsilon) \subset E$. Let $\varphi_i : U \to R$ continuous vanish outside $f^{-1}(B_i)$ and be positive on B_i; we may assume $\sum_i \varphi_i \equiv 1$ on U. Let $Y_\epsilon \subset E$ be the space spanned by the y_i, and define:

$$f_\epsilon : U \to Y_\epsilon \quad f_\epsilon(x) = \sum_i \varphi_i(x) y_i.$$

9. If $f_n \to f$ pointwise in X and $\mathcal{F} = \{f_1, f_2, \ldots\} \subset C(X)$ is equicontinuous, then $f \in C(X)$ and $f_n \to f$ u.o.c.

10. (i) Exhibit a countable dense subset of $\mathcal{F}(I, I)$, with the pointwise topology ($I = [0, 1]$).
(ii) Is $\mathcal{F}(I, I)$ separable with the topology of uniform convergence?

11. (Dini) If $f_1 \leq f_2 \leq f_3 \leq \ldots$ is an increasing sequence of functions in $C(X)$ converging pointwise to $f \in C(X)$, then$f_n \to f$ u.o.c. (*Hint:* ETS the set $\{f_1, f_2, \ldots\}$ is equicontinuous.)

12. If X is any space and M is complete metric, let $\mathcal{F} = \{f_n\}_{n \geq 1} \subset C(X; M)$ be a countable equicontinuous set. If $f_n(x)$ converges for all x in a dense subset $D \subset X$, show that f_n converges u.o.c in X.

13. Given $f : R \to R$ an arbitrary function, consider the sequence of translates $f_n(x) = f(x + n)$, $n \geq 1$. Then f_n converges uniformly on $[0, \infty)$ to the constant function L if, and only if, $\lim_{x \to \infty} f(x) = L$.

14. If each $f_n : X \to E$ (X metric, E Banach) is uniformly continuous on X and $f_n \to f$ uniformly on X, then f is uniformly continuous on X.(X is a metric space.)

15. There is no sequence of polynomials converging either to $1/x$ or to $\sin(1/x)$ uniformly on the open interval $(0, 1)$.

16. A monotone sequence of real-valued functions is uniformly convergent provided it has a subsequence with this property.

17. (i) If a sequence of real-valued monotone functions (with domain R) converges pointwise to a continuous function on an interval $I \subset R$, then the convergence is uniform on each compact subset of I.

(ii) On the set of homeomorphisms of the real line, the pointwise topology and the u.o.c. topology coincide.

18. If $\lim f_n(c) = L$ exists (for some $c \in R$, where $f_n : I \to R$ and $I \subset R$ is an interval containing c) and the sequence of first derivatives (f_n') converges to 0 uniformly on I, then $f_n \to L$ uniformly on each compact subset of I.*Example:* $f_n(x) = \sin(\frac{x}{n})$.

19. A sequence of polynomials of degree $\leq k$, uniformly bounded in a compact interval, is equicontinuous on this interval.

20. Let X, Y be spaces with X locally compact Hausdorff, and let $C(X,Y)$ have the compact-open topology, Show that the evaluation map:

$$e : X \times C(X,Y) \to Y, \qquad e(x,y) = f(x)$$

is continuous.

21. Let (Y, d_Y) be a complete metric space, X a set. The uniform metric on Y^X associated to d_Y is:

$$d(f,g) = \sup_X \min\{d_Y(f(x), g(x)), 1\}.$$

(i) Show that (Y^X, d) is complete.

(ii) If X is a topological space, (Y, d_Y) a complete metric space, show that the set $C(X,Y)$ of continuous functions, with the uniform metric, is a complete metric space.

22. (i) Let $\mathbb{R}^{\mathbb{N}}$ be given the product topology. Prove that any compact subset $K \subset \mathbb{R}^{\mathbb{N}}$ has empty interior.

(ii) Let X be the space $[0,1]^{\mathbb{N}}$ with the uniform topology. Show that X is not locally compact.