REVIEW LIST 8: TOPOLOGICAL APPLICATIONS OF DIFFERENTIAL TECHNIQUES (Version date: 7/2/2021)

1. Manifolds with boundary.

1. Let M be a smooth manifold without boundary, $f : M \to \mathbb{R}$ a smooth function with 0 as a regular value. Prove that $\{x \in M; f(x) \geq 0\}$ is a manifold with boundary, and the boundary is $f^{-1}(0)$. *Hint: local form of submersions.*

2. If $U \subset R^k$ and $V \subset R^k_+$ (upper half space) are neighborhoods of 0, there is no diffeomorphism from U to. V.

3. If $f : X \to Y$ is a diffeomorphism of manifolds with boundary, then the restriction ∂f of f to the boundary maps ∂X diffeomorphically to ∂Y.

4. Use Sard's theorem for manifolds without boundary to prove the following version for manifolds with boundary X:

If $f : X \to Y$ is a smooth map (Y a smooth manifold without boundary), then almost every $q \in Y$ is a regular value for both f and the restriction of f to ∂X, $\partial f : \partial X \to Y$.

Classification theorem. Every compact, connected, one-dimensional manifold is diffeomorphic to [0,1] or to S^1.

5. If X is a compact manifold with boundary, there is no smooth map $X \to \partial X$ that is the identity on ∂X (that is, ∂X is not a smooth retract of X.)

6. (Brouwer fixed-point theorem.) (i) Let $f : B^n \to B^n$ be a smooth map, where B^n is the closed unit ball in R^n, a manifold with boundary. Prove that f has a fixed point. (By contradiction: if f has no fixed point, use a chord argument to find a smooth retraction from f to its boundary.)

(ii) Show by an example that there may be no fixed point in the interior of B^n.

(iii) Use Stone-Weierstrass approximation and the smooth version to prove the theorem for continuous maps.

(iv) Prove the following theorem of Frobenius: any $n \times n$ matrix A with positive entries has a positive eigenvalue. *Hint:* Use Brouwer's fixed point theorem for the map induced by A on the $n-1$ dimensional simplex

$$\Sigma = \{x = (x_1, \ldots x_n) \in R^n; x_i \geq 0, \sum_i x_i = 1\}.$$

2. Transversality theorems [G-P, ch. 2].

Generic transversality theorem (parametric). Let $F : X \times S \to Y$ be a smooth map of manifolds (X with boundary, Y and S without boundary.) Let $F : X \times S \to Y$ be a smooth map (we think of $F = (f_s)_{s \in S}$, a parametrized family of smooth maps from X to Y.) Suppose F and ∂F are both transversal to a fixed closed submanifold $Z \subset Y$ (without boundary.) Then for almost every $s \in S$, f_s and ∂f_s are transversal to Z.

Informally: “Maps in a parametrized family are transversal (to a fixed submanifold) for generic parameter values”.

3. Transversality homotopy theorem. $f : X \to Y$ smooth, $Z \subset Y$ closed submanifold (Y, Z without boundary.) Then arbitrarily close to f (in some C^r topology if X is compact, or Whitney W^r topology in general) there exists $g : X \to Y$ homotopic to f, so that g and ∂g are transversal to Z.

Informally: “Given a smooth map of manifolds, there exists a nearby smooth map in the same homotopy class, transversal to a given submanifold.”

Transversality extension theorem. Given $f : X \to Y$ smooth, if $\partial f : \partial X \to Y$ is transversal to a submanifold $Z \subset Y$ (both Z and Y without boundary), one may find, arbitrarily close to f, an extension $g : X \to Y$ of ∂f, transversal to Z and homotopic to f.

Informally: “If a boundary map transversal to a submanifold extends to the interior, there exists a nearby, homotopic extension which is also transversal.”

4. Mod 2 intersection and mod 2 degree.

7. (i) Let $f : X \to Y$ be a smooth map transversal to a submanifold $Z \subset Y$ (X compact.). State the condition on dimensions under which the mod 2 intersection number $I_2(f; Z)$ is defined, and give its definition.

(ii) Prove that if $f_0, f_1 : X \to Y$ are transversal to $Z \subset Y$ and homotopic, then $I_2(f_0; Z) = I_2(f_1; Z)$. (If the dimension condition holds.)

(iii) Define $I_2(g, Z)$ for a general smooth map $g : X \to Y$ (not necessarily transversal to Z.)

8. Let $f : X \to Y$ be a smooth map, where X is compact, Y is connected and $dim(X) = dim(Y)$. Prove that $card(f^{-1}(q))$ is constant mod 2, for all *regular values* $y \in Y$ (this is the mod 2 degree of f, $deg_2(f) = I_2(f, \{y\})$. (*Hint:* use the fact f is a local diffeomorphism over a regular value, hence the cardinality of the preimage is locally constant.)

9. Prove there exists a complex number z such that:

$$z^7 + \cos(|z|^2)(1 + 93z^4) = 0.$$

Hint: Consider the map given by this expression, restricted to the circle $|z|^2 = \pi/2$, with values in another circle.

10. if X is compact, $Z \subset Y$ a closed submanifold with $codim_Y(Z) = dim(X)$ and $f : X \to Y$ is homotopic to a constant, then $I_2(f, Z) = 0$ (except possibly if $dim(X) = 0$). *Hint:* Show f is homotopic to a constant $y \notin Z$.

11. (i) if X is compact, $Z \subset Y$ a closed submanifold with $codim_Y(Z) = dim(X)$ and Y is contractible, then for any $f : X \to Y$, $I_2(f, Z) = 0$.
(ii) No compact manifold without boundary (other than a single point) can be contractible. (*Hint:* consider the identity map.)

12. (i) Let $f : S^1 \to S^1$ be a smooth map. Prove that exists $g : R \to R$ smooth so that $f(e^{it}) = e^{ig(t)}$, and $q \in \mathbb{Z}$ so that $g(2\pi) = g(0) + 2\pi q$.

(ii) Prove that $deg_2(f) = q \bmod 2$.

13. Suppose X is a compact manifold, $dim(X) < k$. Then given $f : X \to S^k$ smooth, and $Z \subset S^k$ closed submanifold with $codim_{S^k} Z = dim(M)$, then $I_2(f, Z) = 0$. *Hint:* By Sard's theorem, there exists $p \notin f(X) \cap Z$.

14. (i) If $f : X \to Y$ has $deg_2(f) \neq 0$, (X compact, $dim(X) = dim(Y)$), then f is surjective.

(ii) If Y is not compact (but X is), then $deg_2(f) = 0$ for any smooth map $f : X \to Y$.

5. Winding number, Jordan-Brouwer separation theorem.

15. Let X be a compact n- manifold with boundary, $f : \partial X \to R^n$ a smooth map, $z \in R^n$ a point not in the image of f (so the 'mod 2 winding number' $w_2(f, z)$ is well-defined.) Suppose f extends smoothly to $F : X \to R^n$, and that z is a regular value for F (this can always be arranged by moving z slightly); in particular, $F^{-1}(z)$ is a finite set in $int(X)$. Prove that $W_2(f, z) = card(F^{-1}(z))$ (mod 2).

16. Let $X \subset R^n$ be a smooth, compact, connected, embedded hypersurface. Given $z \in R^n \setminus X$, let r be a ray emanating from z that is transversal to X. Show that z is inside X (that is, in the bounded component of the complement) if and only if r intersects X at an odd number of points.

6. Borsuk-Ulam theorem. Odd maps of spheres have nonzero mod 2 degree.

Equivalently, maps $f : S^k \to R^{k+1} \setminus \{0\}$ with odd symmetry $f(-x) = -f(x)$ have nonzero mod 2 winding number $w_2(f, 0)$.

Proof. This is proved by induction, and the induction step follows from two observations. Let $g : S^{k-1} \to R^{k+1} \setminus \{0\}$ be the restriction of f to the equatorial sphere. Let $a \in S^k$ be a common regular value of the normalizations $\hat{f}, \hat{g}$, with values in S^k, and let $l = \mathbb{R}a$, the one-dimensional subspace of R^{k+1} spanned by a. So $a \notin im(g)$ and $f \pitchfork l$. Let f_+ be the restriction of f to S^k_+, the closed upper hemisphere.

A. Use the fact f has odd symmetry to show that $w_2(f, 0) = card(f_+^{-1}(l))$ (mod 2).

Let $\pi : R^{k+1} \to H$ be the orthogonal projection onto the k-dimensnonal subspace H orthogonal to l. Note $\pi \circ g$ maps S^{k-1} to $H \setminus \{0\}$ and has odd symmetry, so we may apply the induction hypothesis to conclude $w_2(\pi \circ g, 0) = 1$.

B. Show that $w_2(\pi \circ g, 0) = card(f_+^{-1}(l))$ (mod 2), concluding the proof. (*Hint:* $\pi \circ f_+ : S^k_+ \to H$ extends $\pi \circ g$ to S^k_+ and problem 15.)

In the proof of Borsuk-Ulam, the following fact is used:

17. Let $\hat{f} : S^k \to S^k$ be the normalization of f, and let r_a be the one-dimensional ray of R^{k+1} defined by $a \in S^k$ (note r_a is a submanifold of $R^{k+1} \setminus$

$\{0\}$, unlike the subspace $l = \mathbb{R}a$. Then a is a regular value for $\hat{f}$ if and only if $f \pitchfork r_a$.

18. (i) If $f : S^k \to R^{k+1} \setminus \{0\}$ is an odd smooth map, then f intersects each line through 0 at least once.

(ii) Any smooth odd map $f : S^k \to R^k$ takes the value $0 \in R^k$ at some point;

(iii) For any $f : S^k \to R^k$ smooth there exists $x \in S^k$ so that $f(-x) = f(x)$.

7. Orientable manifolds/oriented double cover.

Theorem: any connected manifold M admits a two-sheeted covering $p : \tilde{M} \to M$ by an oriented manifold $\tilde{M}$. If $p(x_1) = p(x_2)$ with $x_1 \neq x_2$, the linear isomorphism $dp(x_2)^{-1} \circ dp(x_1) : T_{x_1}\tilde{M} \to T_{x_2}\tilde{M}$ is orientation-reversing.

19. Let $f : X \to Y$ be a diffeomorphism of connected oriented manifolds. Show that if $df(x)$ preserves orientation at one point x, then f preserves orientation globally.

20. Any compact hypersurface in euclidean space is orientable. *Hint:* Jordan-Brouwer separation theorem.

21. Every simply-connected manifold X is orientable.

22. Let M be a differentiable manifold, orientable or not. Then the tangent bundle TM is orientable.

23. Let X be an oriented manifold, $f : X \to R$ a smooth function with $0 \in R$ as a regular value. Prove that the submanifold $Z = f^{-1}(0)$ of X is orientable.

24. Let $f : X \to Y$ be a local diffeomorphism. If Y is orientable, then so is X.

8. Hopf degree theorem.

25. (i) Let $f : U \to R^k$ be a smooth map, $U \subset R^k$ open, $x \in U$ a regular point. If B is a small ball centered at x and $\partial f : \partial B \to R^k$ is the restriction of f to ∂B, prove that the oriented winding number $W(\partial f, z) = \pm 1$, according to whether f preserves or reverses orientation at x.

(ii) Let $f : W \to R^k$ be a smooth map, $W \subset R^k$ a compact k-dimensional manifold with boundary, $\partial f : \partial W \to R^k$ the restriction of f. Let $z \in R^k$ be a regular value of f that has no preimages on ∂W. Prove that the number of preimages of z in W, counted with orientation, equals the oriented winding number $W(\partial f, z)$,

26. (i) Let $B \subset R^k$ be a closed ball, $f : R^k \setminus int(B) \to Y$ a smooth map (Y is any manifold). Show that if the restriction $\partial f : \partial B \to Y$ is homotopic to a constant, f extends to a continuous map from R^k to Y.

(ii) Show that the extension in part (i) can be taken to be a smooth map. (Assume the homotopy in (i) is smooth.)

27. Let $W \subset R^N$ be a compact submanifold without boundary, $f : W \to$

R^{k+1} any smooth map. Show that f extends to a smooth map from R^N to R^{k+1}. (In particular, this is also true if $W \subset R^N$ is a compact submanifold with boundary.)

Hint: Consider a tubular neighborhood of W, and a suitable 'bump function' supported in it.

Recall the *Tubular Neighborhood Theorem.* Let $X \subset R^k$ be an embedded compact submanifold (without boundary.) Then there exists $\epsilon > 0$ so that the set:

$$\mathcal{N}_\epsilon = \{y \in R^k | d(y, X) \leq \epsilon\}$$

is a $(k-1)$-dimensional manifold with boundary in R^k. The closest-point projection $p : \mathcal{N}_\epsilon \to X$ is well-defined and a smooth submersion onto X.

Special case of Hopf's theorem for spheres. Any smooth map $f : S^k \to S^k$ of degree zero is homotopic to a constant.

This is proved by induction on k. The case $k = 1$ is included in the following problem:

28 (i) Recall (problem 12) given any map $f : S^1 \to S^1$ there exists a map $g : R \to R$ so that $f(e^{it}) = e^{ig(t)}$, and $g(t + 2\pi) = g(t) + 2\pi q$ for some $q \in \mathbb{Z}$. Show $deg(f) = q$ (oriented degree.)

(ii) Use part (i) to show that two maps from S^1 to itself with the same degree are homotopic.

The key step in the induction argument is the following.

29. Let $f : R^k \to R^k$ be a smooth map with 0 as a regular value. Suppose $f^{-1}(0)$ is finite, with algebraic cardinality zero (when counted with the orientation sign.) Assuming the special case of Hopf's theorem for S^{k-1}, prove there exists a mapping $g : R^k \to R^k \setminus \{0\}$, so that $g \equiv f$ outside of a compact set. *Hint:* Consider a ball $B \subset R^k$ containing the preimage of 0. What is the winding number of the restriction of f to ∂B?

Remark. The same would work for maps $f : S^k \to R^k$: under the given hypotheses, they can be changed inside a spherical cap, so that the modified map avoids 0. To achieve the induction step one just has to start from $f : S^k \to S^k$ of degree 0 and modify f to a map $g : S^k \to S^k \setminus \{point\}$ in the same homotopy class. (Since g is homotopic to a constant map.)

30. Assuming the fact that maps from S^k to S^k of degree 0 are homotopic to a constant, prove the *general extension theorem:* Let W be a $(k+1)$-dimensional compact, connected, oriented manifold with boundary $\partial W = X$. If $f : X \to S^k \subset R^{k+1}$ has degree 0, then f admits a smooth extension $F : W \to S^k$.

The general Hopf theorem (two maps with the same degree from a compact oriented manifold (without boundary) to a sphere of the same dimension are homotopic) follows easily from this.

Hint. Take an arbitrary extension $F_1 : W \to R^{k+1}$ of f; we may suppose (by the tranversality extension theorem) 0 is a regular value of F_1. Consider a ball

B in W containing the preimage $F_1^{-1}(0)$, a finite set. Show that the winding number of F_1 on ∂B (with respect to $0 \in R^{k+1}$) is zero; this implies $F_1/|F_1|$ restricted to ∂B is homotopic to a constant.

31. Show that any (smooth) map $S^n \to S^n$ with degree different from $(-1)^{n+1}$ must have a fixed point.

32. (i) Show that the degree of the antipodal map of S^n is $(-1)^{n+1}$. Hence the antipodal map is not homotopic to the identity if n is even.

(ii) Show that S^n admits a vector field without singularities if n is odd.

(iii) Show that if S^n admits a vector field without singularities, then the antipodal map is homotopic to the identity. Hence this can't happen if n is even. (*Hint:* Use the vector field to find a map of S^n which is both homotopic to the identity and has no fixed points (hence is homotopic to the antipodal map–prove this too.)

9. Indices of isolated singularities of vector fields/Poincaré-Hopf theorem.

33. Let V be a vector field with isolated zeros in R^k, W a compact k-dimensional submanifold of R^k with boundary. Assume V is never zero on ∂W. Prove that the sum of indices of V at its zeros equals the degree of the map $\frac{V}{||V||} : \partial W \to S^k$.

Let X be a vector field on R^n with finitely many singularities so that the sum of the indices of its singularities is zero. Show that there exists a vector field Y on R^n with no singularities, equal to X outside a compact set.

(This is the first step in the proof that compact oriented manifolds with zero Euler characteristic admit non-vanishing vector fields.)

34. Prove that the Euler characteristic of S^n is 2 if n is even, 0 if n is odd. (*Hint:* If n is even, define a south-pole/north-pole vector field with only two singularities, and compute their indices; this works in n is odd too, or recall there is a vector field without singularities in this case. Or use a triangulation.)

35. Compute the Euler characteristic of a compact oriented surface of genus g (a sphere with g handles attached), by suitably modifying a triangulation of the sphere, and attaching a triangulation of the handles. (Answer: 2-2g.)