TOPOLOGY PRELIM TOPICS

The following topics were included in the sequence M561-562 in 2020-21. The students had prior exposure to topology and metric spaces at the undergraduate level. Suitable undergraduate texts are, for instance, M. A. Armstrong, *Basic Topology* (Springer 1983) or T. W. Gamelin and R.E. Greene, *Introduction to Topology* (Dover 1999).

Part I: General Topology.

1. Basis of a topology, sub-basis, 1st countable, 2nd countable and separable spaces. Interior and closure, limits, Hausdorff spaces. Continuous maps, homeomorphisms and embeddings. Subspaces (induced topology), product spaces. Open maps, closed maps. Quotient topology and quotient maps.

2. Metric and metrizable spaces. Equivalent metrics and quasi-isometry. Completeness, metric completion. G_δ sets, Baire's theorem, Baire spaces. Heine-Borel property.

3. Existence of continuous functions: extension to the closure, regular spaces, normal spaces. Urysohn's lemma, Tietze extension theorem. Hilbert cube and Urysohn metrization.

4. Compact spaces, sequential compactness. Locally compact spaces. Compactness in metric spaces. Alexandroff compactification. σ-compact spaces. Proper maps. Tychonoff's theorem. Compactifications, Stone-Cech.

5. Connected and path connected spaces. Cantor sets, Cantor spaces (Moore-Kline theorem).

6. Equicontinuity, Arzelà-Ascoli theorem. Stone-Weierstrass theorem. Topologies in spaces of maps; compact-open topology.

References:

J. Munkres, *Topology.* Ch 2 to Ch 5, Ch 7, sect. 48

S. Willard, *General Topology* (Dover 2004). Ch 2, Ch 3, Ch. 5 to Ch. 8, Ch. 10

Part II: Differential Topology.

7. Differentiable manifolds, differentiable maps, diffeomorphisms. Tangent space, tangent bundle, differential of a map. Submanifolds: images of embeddings, preimage of a regular value. Transversality. Paracompactness.

Partitions of unity–existence and applications: extension of maps to $\mathbb{R}^n$, embeddings into $\mathbb{R}^n$, Riemannian metrics. Manifolds with boundary.

8. The C^1 and Whitney topologies for differentiable maps. Stability of immersions, submersions, embeddings, diffeomorphisms. Sard's theorem. Whitney's immersion and embedding theorems. Generic transversality theorem, transversality homotopy theorem, extension of transversal maps.

9. Mod 2 intersection theory, mod 2 degree. Brouwer fixed-point theorem. Winding numbers, Jordan-Brouwer separation theorem, Borsuk-Ulam theorem. Orientable manifolds, oriented double cover. Brouwer degree. Hopf degree theorem for maps to the n-sphere. Vector fields with isolated singularities, Poincaré-Hopf theorem.

References:

V. Guillemin, A. Pollack, *Differential Topology* (AMS Chelsea, 2014): Ch 1 to Ch. 3 (except Ch.3, sect 4.)

J. Milnor, *Topology from the Differentiable Viewpoint* (U. Virginia Press, 1965): Chapters 1 through 6.

Part III: Fundamental Group and Covering Spaces.

10. Homotopic maps; contractible spaces, homotopy equivalence. Retracts and deformation retracts. Homotopy to the identity or the antipodal map on spheres. Homotopy of paths and of based loops: fundamental group. Free homotopy of closed curves. Fundamental group of the circle. Induced map on fundamental groups. n-spheres ($n \geq 2$) are simply-connected.

11. Covering maps. Unique lifting of paths and homotopies. Contrast with surjective local homeomorphisms; proper coverings. The fundamental lifting theorem. The subgroup of π_1 induced by a covering map; regular coverings. Covering homomorphisms and automorphisms.

12. Fundamental group of a union. Seifert-van Kampen theorem and applications; adjoining a cell, wedges of spaces, fundamental group of surfaces. Properly discontinuous actions and regular coverings. Existence of the universal cover.

References: J. Munkres, *Topology*, Chapters 9, 11, 13.

E. Lima, *Fundamental Groups and Covering Spaces* (A.K. Peters, 2003)