PROBLEM SET 4–SUMMER 2023–MANIFOLDS

1. Show that a continuous proper map $f: M \to N$ (differentiable manifolds) is a closed map; use this to prove that an injective proper immersion $f: M \to N$ is an embedding.

2. Let $f: M \to N$ be a smooth map, $q \in N$ a regular value of f, $S = f^{-1}(q)$ (assumed non-empty), $x \in S$. Prove that for the tangent space we have:

$$T_x S = Ker(df(x)).$$

Hint: prove it for smooth maps of euclidean space first.

3. Let $f: M \to N$ be a C^k map, transversal to a submanifold $S \subset N$. Let $V \subset M$ be the submanifold $V = f^{-1}(S)$, assumed non-empty. Then if $p \in V$, prove the tangent space T_pV is:

$$T_pV = (df(p))^{-1}[T_{f(p)}S],$$

the preimage under the differential of f.

4. let X and Z be transversal submanifolds of the manifold Y. (Recall this means the inclusion map $i: X \to Y$ is transversal to Z.) Prove that $X \cap Z$ is a submanifold of Y (of what dimension?), and that if $x \in X \cap Z$:

$$T_x(X \cap Z) = T_xX \cap T_xZ.$$

5. Prove: The space of rank one 2×2 matrices (that is, nonzero matrices with determinant zero) is a 3-dimensional submanifold of $M_2 = \mathbb{R}^4$. Find its tangent space at

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (1)$$

6. Let M_n, S_n be the vector spaces of $n \times n$ matrices (resp. $n \times n$ symmetric matrices), and let $f: M_n \to S_n$ be the smooth map $f(A) = AA^t$ (the superscript t means 'transpose'.)

(i) Compute the differential of f at an arbitrary matrix A.

(ii) Show that the identity matrix I_n is a regular value of f, and therefore the orthogonal group $O(n)$ is a manifold (compute its dimension).

(iii) Compute the tangent spaces $T_{I_n}O(n)$ and $T_AO(n)$ for $A \in O(n)$.

(iv) Show that $O(n)$ is compact.

7. (a) Let V be a finite-dimensional real vector space, $\Delta \subset V \times V$ the diagonal. For a linear map $A \in \mathcal{L}(V)$, show that the graph $W = \{(v, Av); v \in V\} \subset V \times V$ is transversal to the diagonal if and only if 1 is not an eigenvalue of A.

(b) *Def.* A smooth map $f: X \to X$ is a *Lefschetz map* if at any fixed point x of f (that is, $f(x) = x$), 1 is not an eigenvalue of the differential $df(x) \in \mathcal{L}(T_xX)$.

Prove that a Lefschetz map of a compact manifold has only finitely many fixed points.

8. (i) Consider the open cover $\{U, V\}$ of the real line, where $U = (-\infty, 1), V = (0, +\infty)$. Show there does not exist a partition of unity with compact supports, strictly subordinate to this cover.

(ii) Let M be a connected smooth manifold. Prove that given $p, q \in M$, there exists a piecewise C^1 path from p to q.

9. *Differentiable Urysohn lemma.* Let M be a smooth manifold, $A, B \subset M$ disjoint closed subsets. Show there exists a smooth (function. $f : M \to [0, 1]$ so that $f \equiv 0$ on A, $f \equiv 0$ on B. *Hint:* smooth partition of unity strictly subordinate to $\{A^c, B^c\}$.

10. On any smooth manifold X there exists a smooth proper function $f : X \to \mathbb{R}$.

Hint: Let $\{U_\alpha\}$ be the family of all precompact open subsets of X, $(\phi_i)_{i\geq 1}$ a subordinate smooth partition of unity. Consider:

$$f(x) = \sum_{i=1}^{\infty} i\phi_i(x).$$

Show that f is well-defined, smooth and proper.

11. Let $f : X \to R^N$ be an injective immersion, where X is a k-dimensional manifold and $N > 2k + 1$. Define the maps:

$$h : X \times X \times R \to R^N, \quad h(x, y, t) = t(f(x) - f(y)).$$

$$g : TX \to R^N, \quad g(x, v) = df(x)[v].$$

(i) Show there exists $a \in R^N$ nonzero which is neither in the image of h nor in the image of g. (*Hint:* Sard's theorem.)

(ii) Show that for such a, if $H \subset R^N$ is the orthogonal complement of the one-dimensional subspace spanned by a and $\pi : R^N \to H$ the orthogonal projection, then $\pi \circ f : X \to H$ is injective.

(iii) Show that $\pi \circ f$ is an immersion.

Conclusion: If the manifold X is compact, X can be embedded into R^{2k+1}.

12. Show that if X is a k-dimensional compact smooth manifold, there exists an immersion $f : X \to R^{2k}$.

13. Prove that if $f : M \to R^n$ is a smooth injective immersion and $\phi : M \to R_+$ is a smooth proper function, then $g(x) = (f(x), \phi(x))$ defines a smooth injective immersion from M to R^{n+1}, which in addition is a proper map.

14. (i) Define 'set of measure 0' in $\mathbb{R}^n$.

(ii) Show that if $A \subset R^n$ has measure zero and $f : R^n \to R^m$ is a locally Lipschitz map (where $m \geq n$), then $f(A)$ has measure zero in R^m.

(iii) Explain why the notion 'set of measure 0' makes sense on differentiable manifolds.

15. (i) Let $f : \mathbb{R} \to \mathbb{R}^2$ be a Lipschitz map (for the euclidean norms.) Prove that $f(\mathbb{R})$ is a null set in $\mathbb{R}^2$. (*Hint:* Do the case of a Lipschitz map $f : [a, b] \to \mathbb{R}^2$ first.)

(ii) Let $Q = \{r_1, r_2, \ldots\}$ be an enumeration of the rational numbers, and let A_{ij} be the open interval with center r_i and length $1/2^{i+j}$, for each $i, j \geq 1$. Show that the set:

$$A = \bigcap_{j\geq 1} \bigcup_{i\geq 1} A_{ij}$$

is a residual subset of the real line, and also a null set.

16. (i) Show that if $k < l$, $\mathbb{R}^k$ has measure zero in $\mathbb{R}^l$.

(ii) Suppose $Z \subset X$ is a submanifold with $dim(Z) < dim(X)$. Prove that Z is a set of measure 0 in X.

17. If $dim(X) < dim(Y)$, the image of any C^1 map $f : X \to$ is a set of measure zero in Y. (Prove this without using Sard's theorem.)

18. Let X be a compact smooth manifold. Every continuous map $f : X \to S^n \subset R^{n+1}$ may be approximated by a smooth map, homotopic to f.

Hint. Assume $X \subset R^N$ (embedded), and use the Stone-Weierstrass theorem for each of the $n+1$ components of f to approximate f by a smooth map $g : X \to R^{n+1}$. Then normalize g, observing that $||g(x)|| > 1 - \epsilon$ if $||f(x) - g(x)|| < \epsilon$.

19. Let $f : M \to R^s$ be a C^1 map, $N \subset R^s$ a submanifold of codimension strictly greater than $dim(M)$, Then for almost every $v \in R^s$ the translated image $f(M) + v$ has empty intersection with N. (That is, the set of $v \in R^s$ for which the intersection is *not* empty has measure zero in R^s.)

20. If $dim(M) < p$, M compact, any C^1 map $f : M \to S^p$ is nullhomotopic.