SOME SOLUTIONS–TOPOLOGY REVIEW, SUMMER 2023.

4 (Problem set 5). The equivalence relation defining X/A is $x \sim x'$ if $x = x'$ or $x, x' \in A$. Let $\bar{x} = p(x) \in X/A$ denote the equivalence class of $x \in X$, where $p : X \to X/A$ is the quotient map. X/A is given the quotient topology: $U \subset X/A$ is open iff $V = p^{-1}(U)$ is open in X. Since $p^{-1}(U)$ is invariant under $\sim$, we see that V is either disjoint from the closed set A, or an open neighborhood of A. Thus the topology in X/A is generated by the images under p of these two types of open sets in X.

(i) If A is contractible, there exists $a_0 \in A$ and a homotopy in A from id_A to the constant map from A to a_0. By Borsuk's homotopy extension theorem (which applies since $X \subset R^n$ is a closed ENR), we find $f_t : X \to X$, $t \in I$, extending that homotopy, with $f_0 = id_X$. Thus $f_t(A) \subset A$ for all $t \in I$ and f_1 maps A to a_0.

(ii) The homotopy (f_t) in (i) induces $h_t : X/A \to X/A$ via $h_t(\bar{x}) = p(f_t(x))$. To see this is well-defined, it suffices to note that:

$$x \sim x', x \neq x' \Rightarrow x, x' \in A \Rightarrow f_t(x), f_t(x') \in A \Rightarrow f_t(x) \sim f_t(x'), \forall t \in I.$$

Since $h_t \circ p = p \circ f_t$ (continuous), h_t is continuous as well. For $t = 1$ we have more :

$$x \sim x', x \neq x' \Rightarrow x, x' \in A \Rightarrow f_1(x) = f_1(x') = a_0.$$

Thus the map $q : X/A \to X$, $q(\bar{x}) = f_1(x)$, is well-defined and continuous.

By definition, $q \circ p = f_1$, which via f_t is homotopic in X to id_X.

We claim that also $p \circ q \simeq id_{X/A}$. Letting $k = p \circ q : X/A \to X/A$, note:

$$k(\bar{x}) = (p \circ q)(\bar{x}) = (p \circ f_1)(x),$$

while $(p \circ f_0)(x) = p(x) = \bar{x}$. So letting $h_t(\bar{x}) = (p \circ f_t)(x)$ we have (as noted above) a well-defined homotopy $h_t : X/A \to X/A, t \in I$, from $h_0 = id_{X/A}$ to $k = h_1 = p \circ q$.

Thus X and X/A are homotopy equivalent spaces.

9. (Problem set 5). (i) Define a family $(h_t)_{t \in I}$ of homeomorphisms of h equal to the identity on ∂B, setting:

$$\text{for } 0 \leq t < 1 : h_t(x) = x, 1-t \leq ||x|| \leq 1; \quad h_t(x) = (1-t)h(\frac{x}{1-t}), 0 \leq ||x|| \leq 1-t.$$

Since h is the identity on ∂B, it follows the two expressions coincide when $||x|| = 1 - t$:

$$(1-t)h(\frac{x}{1-t}) = ||x||h(\frac{x}{||x||}) = ||x||\frac{x}{||x||} = x.$$

For $t = 1$, we set $h_1(x) = x, x \in B$. Clearly all the h_t are homeomorphisms of the closed ball.

We have to check continuity at $t = 1$, but since a possible issue only occurs when $||x|| < 1 - t$, it is enough to check continuity at $x = 0, t = 1$, where $h_1(0) = 0$. But if $||x|| < 1 - t$, we have:

$$||h_t(x)|| = (1-t)||h(\frac{x}{1-t})|| \leq (1-t)M,$$

where $M = max\{||h(x)||; x \in B\} \leq 1$, so $||h_t(x)||$ is small when t is near 1. (If $||x|| \geq 1 - t$ with $||x||$ small, recall $h_t(x) = x$.)

Part (ii) of the problem follows easily from part (i), by composition. (Check this explicitly.)

1(iii) (Problem set 6). Let $p : \mathbb{C} \to \mathbb{C}^*$ be the exponential cover, $p(z) = e^z$ (with deckgroup $\mathbb{Z}$, generated by $z \mapsto z + 2\pi i$.) Note p maps the open left halfplane $H = \{z; Re(z) < 0\}$ onto the punctured open unit disk $D^* \subset \mathbb{C}^*$.

To see this covering map p is not a closed map, consider the curve $\Gamma \subset H$ parametrized over $\mathbb{R}$ by:

$$t \mapsto -e^t + it \in \mathbb{C}, t \in \mathbb{R}.$$

Note Γ is contained in H and properly embedded in $\mathbb{C}$, and hence is a closed subset of $\mathbb{C}$. The image of Γ under p is the curve in D^* parametrized by:

$$t \mapsto e^{-e^t} \cdot e^{it}, t \in \mathbb{R}.$$

This (injectively immersed) curve $p(\Gamma)$ spirals towards $0 \in \mathbb{C}$ as $t \to +\infty$, and towards $S^1 = \partial D^* \setminus \{0\}$ as $t \to -\infty$, and hence is not a closed subset of $\mathbb{C}^*$. Thus p is not a closed map.

6 (problem set 6). We claim that X is homotopy equivalent to $S^2 \vee S^1$, hence by S-vK $\pi_1(X)$ is isomorphic to $\mathbb{Z}$.

Consider a closed square region Q, with boundary defined by the vertices a, b, c, d (counterclockwise). Let e be the midpoint of the edge ad, and $T \subset Q$ the closed triangular region with vertices b, c, e. Attach Q to S^2 by mapping the edge ad homeomorphically onto an arc of meridian on S^2 (so this arc equals the intersection of Q and S^2, in the attachment space Y). Note the following:

(i) X is homeomorphic to the subspace X_1 of Y consisting of S^2 and the edges $[ab], [bc], [cd]$ of Q, attached to S^2 at a and d;

(ii) $S^2 \vee S^1$ is homeomorphic to the subspace X_2 of Y consisting of S^2 and the edges $[eb], [bc], [ce]$ of T, attached to S^2 at e;

(iii) Fix a point q_0 in the interior of T. Then X_1 is a deformation retract of $Y \setminus \{q_0\}$ (via 'radial map from q_0'). Likewise, X_2 is a deformation retract of $Y \setminus \{q_0\}$. Thus X_1 and X_2 are homotopy equivalent, and X and $S^2 \vee S^1$ are also homotopy equivalent.

6(iii), Problem set 7. Let $f : B \to B$ be continuous. Proceeding by contradiction, suppose f has no fixed points in B. Then $||f(x) - x|| \geq c > 0$ for some $c > 0$ and all $x \in B$. Let $\epsilon > 0$ be arbitrary. By Stone-Weierstrass, we may find $g : B \to R^n$ smooth, so that $||g(x) - f(x)|| < \epsilon$ for all $x \in B$. In particular, $||g(x)|| < 1 + \epsilon$ in B, so defining $\hat{g}(x) = \frac{g(x)}{1+\epsilon}$, we have $\hat{g} : B \to B$ smooth, as well as:

$$||f(x) - \hat{g}(x)|| < \epsilon + ||g(x)||(1 - \frac{1}{1+\epsilon}) \leq 2\epsilon.$$

So $||\hat{g}(x) - x|| \geq ||f(x) - x|| - 2\epsilon \geq c - 2\epsilon \quad \forall x \in B$, contradicting the existence of fixed points in the smooth case, if $\epsilon < c/2$.

6(iv) Note Σ is homeomorphic to the $(n-1)$-closed ball, and that A preserves the 'closed positive cone' K of R^n, $K = \{v \in R^n; v_i \geq 0 \forall i\}$. For $v \in K$, let $\sigma(v) \geq 0$ be the sum of the coordinates. Then:

$$f(x) = \frac{Ax}{\sigma(Ax)}$$

defines a continuous map from Σ to itself (note $\sigma(Ax) \neq 0$ for $x \in \Sigma$, since A has positive entries). Letting $x_0 \in \Sigma$ be a fixed point of f, we have:

$$Ax_0 = \sigma(Ax_0)x_0, \quad x_0 \neq 0,$$

so x_0 is an eigenvector of A, with positive eigenvalue $\sigma(Ax_0)$.

12, Problem set 7. (i) Let $p : \mathbb{R} \to S^1$, $p(t) = e^{it}$, be the standard exponential covering. Since $\mathbb{R}$ is contractible, by the fundamental lifting criterion the map $f \circ p : \mathbb{R} \to S^1$ lifts over p, so there exists $g : \mathbb{R} \to \mathbb{R}$ continuous, satisfying $p \circ g = f \circ p$. We have:

$$e^{ig(t+2\pi)} = f(e^{i(t+2\pi)}) = f(e^{it}) = e^{ig(t)} \quad \forall t,$$

so there exists $q \in \mathbb{Z}$ so that $g(t + 2\pi) = g(t) + 2\pi q$, for all $t \in \mathbb{R}$.

(ii) Fix the $q \in \mathbb{Z}$ obtained for f in part (i) and consider the basic example, the map of S^1 given by $f_0(z) = z^q$, or $f_0(e^{it}) = e^{iqt}$. Clearly

$g_0(t) = qt$ in this case. Composing f with a rotation (which doesn't change its mod 2 degree) we may assume $f(1) = 1$ and $g(0) = 0, g(2\pi) = 2\pi q = g_0(2\pi)$.

We see that g is homotopic to g_0 on $[0, 2\pi]$ with fixed endpoints, via $g_s(t) = sg(t) + (1-s)g_0(t), s \in [0,1]$, and also on $\mathbb{R}$, extending g_s via $g_s(t + 2\pi) = g_s(t) + 2\pi q$. This periodicity shows $g_s : \mathbb{R} \to \mathbb{R}$ induces a homotopy $f_s : S^1 \to S^1$ from f_0 to f (via $f_s(e^{it}) = e^{ig_s(t)}$). By homotopy invariance of the degree, $deg_2(f) = deg_2(f_0)$, and the latter is seen to equal q mod 2, by counting preimages of an arbitrary point on S^1.

17, Problem set 7. In general, let X be any smooth manifold, $f : X \to R^{k+1} \setminus \{0\}$ a smooth map, $\hat{f} : X \to S^k$ the normalization of f. For $a \in S^k$, denote by $r_a \subset R^{k+1} \setminus \{0\}$ and l_a the open ray and the one-dimensional subspace of R^{k+1} defined by a. By definition, $a \in S^k$ is a regular value for $\hat{f}$ iff:

$$d\hat{f}_x : T_xX \to T_aS^k = l_a^\perp \text{ is onto, } \forall x \in \hat{f}^{-1}(a); \qquad (1)$$

while $f \pitchfork r_a$ iff:

$$df_x : T_xX \to R^{k+1} \text{ satisfies } df_x[T_xX] + l_a = R^{k+1}, \text{ if } f(x) \in r_a. \qquad (2)$$

It is clear that $f(x) \in r_a$ iff $\hat{f}(x) = a$. To show equivalence of the two statements, recall the 'Calculus fact': for $x \in X$, $v \in T_xX$ and $a = \hat{f}(x)$:

$$d\hat{f}_x[v] = \frac{1}{||f(x)||}(df_x[v] - \langle df_x[v], a\rangle a) = \frac{1}{||f(x)||} proj_{l_a^\perp}(df_x[v]) \in T_aS^k, \qquad (3)$$

where $proj_{l_a^\perp}$ denotes the orthogonal projection from R^{k+1} onto the orthogonal complement of l_a (with respect to the standard inner product in R^{k+1}).

Assuming (1), we prove (2). Let $w \in R^{k+1}$, write $w = w_1 + w_2$, with $w_1 \in l_a, w_2 \in l_a^\perp$. From (1), $w_2 = d\hat{f}_x[v]$, for some $v \in T_xX$. And then (3) shows $df_x[v_1] = d\hat{f}_x[v] + w_3$, with $w_3 \in l_a$ and $v_1 = ||f(x)||^{-1}v \in T_xX$. Thus $w = df_x[v_1] + (w_1 - w_3)$, proving (2).

Conversely, assuming (2), we prove (1). Let $w \in l_a^\perp = T_aS^k$. From (2), $w = df_x[v]$ for some $v \in T_xS^k$ (where $\hat{f}(x) = a$; clearly w has no component in l_a.) But then (3), combined with the fact $df_x[v] \in l_a^\perp$, imply $d\hat{f}_x[||f(x)||v] = df_x[v] = w$, proving (1).

21, problem set 8. This is a direct consequence of the fact that any manifold X admits an oriented double cover $p : \tilde{X} \to X$, unique up to automorphisms (Proposition 8.7 in [Lima, p. 196]); and that this cover $\tilde{X}$

is disconnected if and only if X is orientable (Proposition 8.5). (You should review those results, Ch. 8 of [Lima].)

When X is simply connected, any covering space of X (with more than one sheet) is disconnected, and the covering map is trivial; thus in this case X is orientable.

22, problem set 8. If $\varphi : U \to U_0 \subset R^n$ is a chart for M, the corresponding chart for $TM_{|U}$ is:

$$\tilde{\varphi} : TM_{|U} \to U_0 \times R^n, \quad \tilde{\varphi}(x, v) = (\varphi(x), d\varphi(x)[v]).$$

Let $\psi : V \to V_0 \subset R^n$ be another chart, with overlapping domain ($U \cap V \neq \emptyset$.) The change of coordinates map is the diffeomorphism:

$$F = \psi \circ \varphi^{-1} : U_1 \to V_1, \quad U_1 \subset U_0, V_1 \subset V_0,$$

and the associated change of coordinates for TM is:

$$\tilde{F} : U_1 \times R^n \to V_1 \times R^n, \quad \tilde{F}(x, v) = (F(x), dF_x[v]).$$

Its differential at a point (x, v) is given by:

$$d\tilde{F}_{(x,v)} : R^n \times R^n \to R^n \times R^n,$$

$$d\tilde{F}_{(x,v)}[(w_1, w_2)] = (dF_x[w_1], d^2F_x[w_1, v] + dF_x[w_2]),$$

where d^2F_x denotes the second differential of F at x. (Note that for fixed x, $\tilde{F}(x, \cdot) = (F(x), dF_x[\cdot])$, where $dF_x \in \mathcal{L}(R^n)$, so the partial differential of $\tilde{F}$ with respect to its second argument at (x, v) is simply $d_2\tilde{F}_{(x,v)} = (0, dF_x)$, independent of v.)

This implies (for example, via matrix expressions for $d\tilde{F}_{(x,v)} \in \mathcal{L}(R^n \times R^n)$ in 'block form'):

$$\det d\tilde{F}_{(x,v)} = \det(dF_x)^2 = (\det dF_x)^2 > 0,$$

so the atlas for TM associated to any atlas of M is automatically orientation-preserving.

24, problem set 8. It is enough to show that, given any $p \in X$, we may find in some neighborhood U of p a set of $n = dim(X) = dim(Y)$ linearly independent vector fields $v_1, \ldots, v_n$ (in this order), so that if two of these neighborhoods U_1, U_2 (of $p_1, p_2 \in M$) intersect, at any point $p \in U_1 \cap U_2$ the transition map from the basis $\{v_1(p), \ldots, v_n(p)\}$ to the basis

$\{w_1(p), \dots w_n(p)\}$ of T_pM (where the w_i are defined in U_2) has positive determinant over $U_1 \cap U_2$.

Given $p \in M$, let $\{e_1, \dots, e_n\}$ be a *positive* set of lin. indep smooth vector fields (with respect to the orientation of Y), defined in some neighborhood $W \subset Y$ of $f(p)$; let $U \subset X$ be a neighborhood of p so that $f(U) \subset W$ and set $v_i(p) = df_p^{-1}[e_i(f(p))]$. Clearly this defines a family of smooth lin.indep. vector fields on U.

To verify the consistency condition, if $p \in U_1 \cap U_2$, let $q = f(p) \in Y$. We have *positive* frames $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_n\}$ in $W_1 \supset f(U_1)$, $W_2 \supset f(U_2)$ (resp.), $q \in W_1 \cap W_2$, and the transition map $A(q)$ from the $e_i(q)$ to the $f_i(q)$) has positive determinant:

$$f_i(q) = \sum_j a_{ij}(q) e_j(q), \quad det(a_{ij})(q) > 0.$$

Since the transition map from the $v_i(p) = (df_p)^{-1}[e_i(q)])$ (in U_1) to the $w_i(p) = (df_p)^{-1}[f_i(q)]$ (in U_2) is also given (at $p \in U_1 \cap U_2$) by $A(q), q = f(p)$, it has positive determinant as well:

$$w_i(p) = (df_p)^{-1}[f_i(q)] = \sum_j a_{ij}(q)(df_p)^{-1}[e_j(q)] = \sum_j a_{ij}(q) v_j(p).$$

Remark: The orientation induced on X by the local diffeomorphism f and an orientation of Y is known as 'pullback orientation'. Note this does not work in the other direction, that is, an orientation on X does not induce one on Y. A simple example is the standard covering map from S^2 to $\mathbb{R}P^2$, a non-orientable manifold (it contains a Möbius strip as an open set.). 'Orientations pull back, but do not push forward'.