TOPOLOGY REVIEW, JUNE 2023: PROBLEM SET 5

1. $f, g : S^n \to S^n$ continuous.

(i) $f(x) \neq -g(x) \forall x \in S^n \Rightarrow f \simeq g$.

(ii) If S^n admits a nonvanishing tangent vector field, then the antipodal map is homotopic to the identity.

2. (i) Define what it means for a closed subset $Y \subset R^n$ to be a 'euclidean neighborhood retract' (ENR).

(ii) Explain why smooth submanifolds of R^n are ENRs.

(iii) Let $Y \subset R^n$ (closed) be an ENR, X a metric space, $A \subset X$ closed. Show that any $f : A \to Y$ continuous extends to a continuous map to Y, defined in an open neighborhood of A in X.

3. (i) $Y \subset R^n$ compact ENR, X metric. Prove there exists $\epsilon > 0$ so that for any $f, g : X \to Y$ continuous, if $|f(x) - g(x)| < \epsilon$ for all $x \in X$, then $f \simeq g$.

(ii) M, N compact smooth manifolds, $f : M \to N$ continuous. Then f is homotopic to a smooth map from M to N.

4. $X \subset R^n$ closed ENR, $A \subset X$ closed, X/A the quotient space obtained by 'crushing A to a point'.

(i) Suppose A is contractible; prove there exists $f : X \to X$ continuous, $f \simeq id_X$, so that $f(A)$ is a point. (*Hint:* Borsuk's homotopy extension theorem.)

(ii) Under the same hypothesis as (i), prove that X and X/A have the same homotopy type.

5. Give an example of a space X and a subspace $A \subset X$ that is a retract, but not a deformation retract of X.

6. Let X be the complement of a point in the two-dimensional torus T^2. Find a subspace $A \subset X$ homeomorphic to the figure-eight space, so that A is a deformation retract of X (and explain why that is the case.)

7. Let $F : B^2 \to B^2$ be continuous (two-dimensional closed unit ball), so that $F(S^1) \subset S^1$. Denote by $f : S^1 \to S^1$ the restriction of F. Prove that either F is onto, or $f \simeq$ const. (possibly both.)

8. Prove that the Möbius strip has the homotopy type of the cylinder $S^1 \times I$, but is not homeomorphic to it.

9. Let $h : B \to B$ (closed unit ball in R^{n+1}) be a homeomorphism, and restrict to id_{S^n} on $\partial B = S^n$. (i) Prove that h is isotopic to id_B (that is, homotopic, and the intermediate maps f_t are homeomorphisms.)

(ii) Prove that if $f, g : B \to B$ (homeomorphisms) map $\partial B = S^n$ to itself and their restrictions to S^n are isotopic on S^n, then f is isotopic to g in B.