Topology prelim, 2023-24: Study guide
General Topology: basis and subbasis for a topology, product topology, quotient topology, separation properties (Hausdorff, regular, normal); countability properties, closure and sequential closure. Compactness and local compactness, compactness in metric spaces (notion of `totally bounded'.) Lebesgue number. Compactness vs. sequential compactness. Sigma-compact locally compact spaces. Paracompactness. Continuity and uniform continuity, extension to the closure. Open maps, closed maps, proper maps. `Tube lemma' for product spaces, `stability of preimages' for closed maps. Embeddings. Complete metric spaces, metric completion. Connected, path connected spaces. Urysohn lemma, Urysohn metrization theorem, Tietze extension theorem. Tychonoff's compactness theorem for product spaces. Baire property and Baire spaces. Continuity sets of functions and pointwise convergence.

This material is (mostly) found in **Munkres**, Chapters 2, 3, 4, especially Ch. 4 (excluding no. 36.) Also no. 43, 45 in Ch.7 and no. 48 in Ch. 8. See also the Appendix "proper maps" to **Lima**.

Topologies in Spaces of Maps: topologies of pointwise convergence, uniform convergence, uniform convergence on compact sets (u.o.c. convergence.) Closure of the space of continuous maps. Compact-open topology. Separation, countability and metrizability properties of function space topologies. Compactly generated spaces. Arzela-Ascoli Theorem. Stone-Weierstrass theorem and applications.

Most of this is found in **Munkres**, no 46, 47 in Ch. 7. For Stone-Weierstrass, see online course notes.

Fundamental group and covering spaces.

Munkres: Chapter 9 (except no. 56): basic facts about homotopy, fund. group, coverings
Ch. 11, no 70: Seifert-v Kampen (no 67, 68, 69 are group theory prerequisites for S-vK)
Ch. 13: Universal covering, existence of coverings, covering automorphisms (all).

L**ima**: Ch. 1, 1.1 to 1.4: homotopic maps, homotopy type, homotooy and extensions, ENR.
Ch. 2, 2.1 to 2.6: fundamental group, free homotopies, induced homomorphism
Ch. 3: 3.1 to 3.3; also 5.1: fund groups of circle, projective spaces, winding number of curves
Ch.6, covering spaces (all)/Ch. 7 (to 7.6) lifting theorem, covering automorphisms, properly discontinuous group actions, existence of coverings.
Ch.8 (all): Orientable manifolds, oriented double cover.

Differentiable manifolds

Main source: **Guillemin-Pollack,** Ch1 to Ch 3 (some topics not found there—see class notes or Milnor's *Topology from the Differentiable Viewpoint*, or Munkres' *Differential Topology.*)

1.Topological and differentiable manifolds: C^r chart, C^r atlas, differentiable maps. Tangent vector at a point, tangent bundle. Sigma-compactness and paracompactness. Diffeomorphisms, immersions, submersions, embeddings, proper embeddings. Submanifolds. Homotopy and stable properties of maps [GP, p.35]. Inverse function theorem, implicit function theorem, local forms of immersions and of submersions. Partition of unity subordinate/strictly subordinate to an open cover. Existence of Riemannian metrics.

2. Critical values and Sard's theorem. Sets of measure zero (or `null sets') in Euclidean space and on manifolds; invariance under locally Lipschitz maps (in particular, under C^1 maps.) transversality. Preimages of regular values, and of submanifolds under transversal maps; their tangent spaces. Manifolds with boundary: topological invariance of boundary points, boundary of the preimage; classification in one dimension. Existence of embeddings of compact manifolds into some Euclidean space; Whitney's embedding and immersion theorems, existence of proper embeddings. Existence of tubular neighborhoods. Parametrized transversality theorem (p.68), transversality homotopy theorem (p.70), Transversal extension theorem (p. 72).

3. Mod 2 intersection theory. Definition of mod 2 intersection number, homotopy invariance, obstruction to extension of maps defined on the boundary. Mod 2 degree of a map. Mod 2 winding number. Jordan-Brower separation theorem (for smooth compact hypersurfaces of Euclidean space.) Borsuk-Ulam theorem.

4. Oriented intersection theory. Orientable manifolds. Preimage orientation and boundary orientation. Oriented intersection number: definition, homotopy invariance. Intersection number of two maps and of two compact submanifolds of a manifold. Self-intersection number of a submanifold in middle dimension (p.115), definition of Euler characteristic in terms of intersection (p. 116). Oriented degree of a map. Lefschetz maps, Lefschetz fixed points. Local Lefschetz index (p. 121, p. 127, p. 129.) Indices of isolated singularities of vector fields: Poincare-Hopf theorem. Hopf degree theorem for maps from a compact k-manifold to the k-sphere. Isotopy lemma (p. 142).

OTHER SOURCES: In addition to the problems in the relevant sections of the texts (Guillemin-Pollack, Lima, Munkres), other problems are found in the homework, tests and handouts for the course this year. And you could also look at the lists of problems included in the Summer 2021 review. Links:

https://web.math.utk.edu/%7Efreire/teaching/PrelimReviewSets2021/Topology_Prelim_Review_2021.html
https://web.math.utk.edu/~freire/teaching/m561f22/m561f22index.html
https://web.math.utk.edu/~freire/teaching/m562s23/m562s23index.html