QUIZ 7: SOLUTIONS

1, $y'' - x^2y' + y = 0$, $y = \sum_{n=0}^{\infty} a_n x^n$: general solution.

$$x^2y' = \sum_{n=1}^{\infty} na_nx^{n+1} = \sum_{n=1}^{\infty}(n-1)a_{n-1}x^n, \quad y'' = 2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n.$$

Leads to:

$$y'' - x^2y' + y = a_0 + 2a_2 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} - (n-1)a_{n-1} + a_n]x^n.$$

Setting $n = 0, 1, 2$ we find:

$$a_2 = -\frac{a_0}{2}, \quad 6a_3 + a_1 = 0, \quad 12a_4 - a_1 + a_2 = 0,$$

and hence:

$$a_3 = -\frac{a_1}{6}, \quad a_4 = \frac{1}{12}(a_1 - a_2) = \frac{1}{12}(a_1 + \frac{a_0}{2}),$$

so the general solution is:

$$y = a_0 + a_1x - \frac{a_0}{2}x^2 - \frac{a_1}{6}x^3 + \frac{1}{12}(a_1 + \frac{a_0}{2})x^4 + \ldots$$

$$= a_0(1 - \frac{x^2}{2} + \frac{x^4}{24} + \ldots) + a_1(x - \frac{x^3}{6} + \frac{x^4}{12} + \ldots),$$

convergent for all $x \in R$.

2. $y' - e^xy = 0$, $\quad y(x) = \sum_{n=0}^{\infty} a_nx^n$, $\quad y(0) = 1$ (so $a_0 = 1$). Expanding $y' = e^xy$ as power series at 0:

$$a_1 + 2a_2x + 3a_3x^2 + \ldots = (1 + x + \frac{x^2}{2} + \ldots)(1 + a_1x + a_2x^2 + \ldots)$$

Comparing the terms of degree 0, 1 and 2 on both sides leads to:

$$a_1 = 1, \quad 2a_2 = 1 + a_1, \quad 3a_3 = \frac{1}{2} + a_1 + a_2,$$

thus $a_2 = 1$ and $a_3 = \frac{5}{6}$, so the solution has the expansion:

$$y(x) = 1 + x + x^2 + \frac{5}{6}x^3 + \ldots,$$

convergent for all $x \in R$.