MATH 231–REVIEW PROBLEMS–NOV 3, 2022.

1. Find the general solution:

$$(i) y'' + 4y = sin2t, \quad y = y(t).$$

$$(ii) x'' - x = t\sin t + t, \quad x = x(t).$$

$$(iii) y'' - \frac{4}{t}y' + \frac{4}{t^2}y = 0, \quad y = y(t), t > 0.$$

$$(iv) y'' - \frac{1}{t-1}y' + \frac{5}{(t-1)^2}y = 0, \quad y = y(t), t > 1.$$

2. Find the general solution to the homogeneous equation:

$$ty'' - (t+1)y' + y = (1-t)^2 = 0, \quad y = y(t), t > 0,$$

given that $y_1(t) = e^t$ is a solution to the corresponding homogeneous equation. (*Hint:* use reduction of order: set $y_2(t) = v(t)y_1(t)$), find a first-order DE for $z = v'$ and solve it, then integrate z to find v, and hence y_2).

3. (i) Show that if $y(t)$ is a solution of $y'' = 2y^3$, the energy function $E(t) = (y')^2 - y^4$ is constant.

(ii) Show that $y(t) = \frac{1}{t-c}$ is a solution for each c, and find the solution with initial conditions $y(0) = 1, y'(0) = -1$, including its domain of definition.

4. (i) An undamped mass-spring oscillator satisfies the differential equation $y'' + 5y = 0$. Show that the energy $E(t) = (y')^2 + 5y^2$ is constant along solutions. With initial conditions $y(0) = 0, y'(0) = 1$, find the amplitude and period of oscillation and sketch its graph.

(ii) Now add a damping term $2y'$, obtaining $y'' + 2y' + 5y = 0$. Show the energy $E(t)$ is *decreasing* along solutions. With the same initial conditions, find the solution and sketch its graph.

5. (i) Solve the IVP $y'' + 3y' + 2y = 0, y(0) = -1, y'(0) = 1$. Sketch the graph of the solution (for $t \in \mathbb{R}$); if it has only one zero and only one critical point, find them.

(ii) Consider oscillations under an external periodic force described by the equation:

$$y'' + y' + y = \sin(\gamma t),$$

where γ is arbitrary. The amplitude of the steady-state solution is $N(\gamma)^{-1/2}$, where $N(\gamma) = (1-\gamma^2)^2 + \gamma^2$. Find γ so that this amplitude is as large as possible.