NOTES ON BAIRE'S THEOREM

Example. A complete metric space (X, d) without isolated points is *uncountable.*

Suppose by contradiction $X = \{x_1, x_2, \ldots\}$. Let $y_1 \neq x_1$ and $0 < r_1 < 1$ be such that $x_1 \notin \bar{B}_{r_1}(y_1)$. Then choose $y_2 \in B_{r_1}(y_1)$ and $r_2 > 0$ so that $y_2 \neq x_2$ and $\bar{B}_{r_2}(y_2) \subset B_{r_1}(y_1)$, with $0 < r_2 < 1/2$. We can do this since X has no isolated points.

Proceeding in this fashion we get a descending chain of closed balls:

$$\bar{B}_{r_1}(x_1) \supset \bar{B}_{r_2}(y_2) \supset \ldots \qquad r_n < \frac{1}{n},$$

so (y_n) is Cauchy, and by completeness $y_n \to y$. But $y \neq x_n$ for all n, contradiction.

Baire's Theorem. Let (G_n) be a countable family of open dense sets in a complete metric space X. Then $\bigcap_{n \geq 1} G_n$ is dense in X (in particular non-empty.)

Informally, a property defined by an open set (within a class X of mathematical objects) is thought of as 'stable'; a property defined by a *dense* subset of X can be thought of as 'generic' (any object in X may be approximated by a sequence of objects with the property).

Example. There exists a function $f \in C[0,1]$ which is not monotone on any interval. In fact the set of such functions is *dense* in $C[0,1]$ (endowed with the sup norm.)

Idea. To see this, let $(I_n)_{n \geq 1}$ be an enumeration of the set of subintervals of $[0,1]$ with endpoints in $\mathbb{Q}$. Let E_n be the set of $f \in C[0,1]$ which are *not* monotone on I_n. The idea is to show that E_n is open and dense, and apply Baire's theorem.

E_n is *open*: if $f \in E_n$, we may find $x < y < z$ in E_n so that $f(x) < f(y)$ and $f(z) < f(y)$ (the other case, f dropping between x and z, is similar.) Then if $||f - g|| < \frac{1}{2} \min\{f(y) - f(x), f(y) - f(z)\}$, it is easy to see that g also fails to be monotone, in the same way as f.

E_n is *dense*: let $f \in C[0,1]$, and say f is monotone increasing on I_n. Pick $x \in I_n$. Given $\epsilon > 0$, we may find $x_- < x < x_+$ very close to x, so that $f(x+)$ and $f(x-)$ are ϵ-close to $f(x)$. Then we can change f slightly in the interval (x_-, x_+) (and nowhere else), to find g continuous and ϵ-close to f in sup norm, so that g is not monotone on this interval (say $g(x_-) > g(x)$ and $g(x_+) > g(x)$), hence not on I_n.

Example. *Uniform Boundedness Theorem.* Let E, F be Banach spaces, and consider a family of linear maps $T_\alpha \in \mathcal{L}(E,F)$, $\alpha \in \Lambda$. If the family is *equibounded* at each $x \in E$ ($||T_\alpha(x)|| < M(x)$ for all $\alpha \in \Lambda$), then it is *uniformly equicontinuous* on E:

$$\sup_{\alpha\in\Lambda} ||T_\alpha|| < \infty.$$

Definition. A Hausdorff topological space X is a *Baire space* if countable intersections of open dense subsets of X are dense.

Theorem. Locally compact Hausdorff topological spaces X are Baire spaces.

Proof. Let $G_1, G_2, \ldots$ be open dense sets. Let $U \subset X$ be open. Then $U \cap G_1 \neq \emptyset$, and $\exists B_1$ open, with compact closure, so that $\bar{B_1} \subset U \cap G_1$. In the same way, we successively find open sets B_n with compact closure, so that $\bar{B_n} \subset B_{n-1} \cap G_n$.

The $\bar{B}_n$ are closed in the compact $\bar{B_1}$ and nested, so $\bigcap_{n\geq 1} \bar{B}_n \neq \emptyset$, and this intersection is contained in $U \cap \bigcap_{n\geq 1} G_n$ (since $\bar{B}_n \subset G_n \forall n$, and $\bar{B}_1 \subset U \cap G_1$. Thus $(\bigcap_{n\geq 1} G_n) \cap U \neq \emptyset$, as we wished to show.

Definition. A subset of a topological space is a G_δ set if it is a countable intersection of open sets.

Example. In a metric space (X,d), any closed set A is a G_δ, since

$$A = \bigcap_{n\geq 1} G_n, \quad G_n = \{x \in X; d(x,A) < \frac{1}{n}\}.$$

Example. The set of rational numbers $\mathbb{Q} \subset \mathbb{R}$ *is not* a G_δ set. If it were, we'd have:

$$\mathbb{Q} = \bigcap_{n\geq 1} G_n,$$

with each G_n open and also dense. (Any open subset of $\mathbb{R}$ intersects $\mathbb{Q}$, hence would intersect each G_n.) But then we can add to the countable family (G_n) of open dense sets the countable family $\{r_n\}^c_{n\geq 1}$ (complement of the one-point sets $\{r_n\}$, where the r_n are an enumeration of $\mathbb{Q}$.) Since each of these sets is open and dense in $\mathbb{R}$, taken together these families would necessarily have nonempty intersection (by Baire's theorem). But clearly the intersection is empty.

What this argument shows is that no countable dense set without isolated points can be a G_δ (in a complete metric space, or a locally compact space.)

Example. Let X be a topological space, Y a complete metric space, $f : X \to Y$ any map. Then the set of continuity C_f of f is a G_δ (which may be empty!)

Indeed, f is continuous at $p \in X$ iff $\forall n \geq 1$ $\exists U$ nbd of p so that $d(f(x), f(y)) < 1/n$ $\forall x, y \in U$. Set:

$$A_n = \{p; \exists U \text{ nbd of } p; d(f(x), f(y)) < \frac{1}{n} \forall x, y \in U\}.$$

Considering the family of open sets of X:

$$\Lambda_n = \{U \text{ open }; d(f(x), f(y)) < \frac{1}{n} \forall x, y \in U\}$$

we have that A_n is the union of this family, an open subset of X:

$$A_n = \bigcup \{U; U \in \Lambda_n\}$$

and clearly:

$$C_f = \bigcap_{n \geq 1} A_n,$$

and hence C_f is a G_δ.

Example. In particular, $\mathbb{Q}$ cannot be the set of continuity of a function from $\mathbb{R}$ to $\mathbb{R}$. But the irrationals $\mathbb{I}$ can be. For example, *Thomae's function*:

$$f(x) = \frac{1}{q}, x = \frac{p}{q} \in \mathbb{Q}, \text{ with } p \in \mathbb{Z}, q \in \mathbb{N} \text{ coprime}; \quad f(x) = 0, x \in \mathbb{I}$$

is continuous exactly at points of $\mathbb{I}$.

Example. Let $f_n : X \to Y$ be continuous (X topological, Y metric.) Suppose $f_n \to f$ pointwise on X. Then each level set $\{x \in X; f(x) = L\}$ of f is a G_δ subset of X. (See the notes for Hw set 10.)