METRIC COMPLETION VIA DISTANCE FUNCTIONS

1. Metrics in spaces of maps.. Let (X, d), (Y, ρ) be metric spaces. The space $F(X, Y)$ of all maps $f : X \to Y$ can be endowed with a metric D, with convergence in D corresponding to uniform convergence over X. Namely, we let:

$$D_1(f, g) = \sup_x \min\{\rho(f(x), g(x)), 1\}.$$

The verification of the triangle inequality follows from the following fact:

Lemma 1. (i) Let a, b, c be positive real numbers satisfying the triangle inequality (in any order). Let $f : R_+ \to R_+$ be an increasing ($f(x) \leq f(y)$ if $x < y$), subadditive function ($f(x + y) \leq f(x) + f(y)$). Then $f(a), f(b), f(c)$ also satisfy the triangle inequality (in any order.)

Proof. Note that we may assume $a \leq b \leq c$ when proving this, and then the non-trivial triangle inequality is $c \leq a + b$; since f is increasing, we also have $f(a) \leq f(b) \leq f(c)$, and just have to show $f(c) \leq f(a) + f(b)$. This follows from monotonicity and subadditivity:

$$f(c) \leq f(a + b) \leq f(a) + f(b).$$

It is easy to check that $f(t) = \min\{t, 1\}, t \geq 0$, satisfies these conditions. (Given two positive reals $x \leq y$, consider the cases $1 \leq x, x \leq 1 \leq y, y \leq 1$.)

All the same, the *min* in the definition of D_1 can be an annoyance, and it is not needed in the space of bounded maps from X to Y:

$$B(X, Y) = \{f : X \to Y | (\exists y_0 \in Y, M > 0)(\forall x \in X)\rho(f(x), y_0) \leq M\}.$$

$$D(f, g) = \sup_x \rho(f(x), g(x)).$$

(The *sup* is well-defined, since both f and g have bounded image). Convergence in the metric D is equivalent to uniform convergence over X.

We have the following important *fact*: $(B(X, Y), D)$ is a complete metric space if and only if (Y, ρ) is complete.

2. Construction of an isometric embedding. In particular, denoting by $B(X)$ the space of bounded functions from X to R, the completeness of R implies $B(X)$ is complete (i.e. a Banach space).

We're going to embed the metric space (M, d) into $(B(M), D)$, and the idea is to map a point x to its distance function $d_x \in C(M)$. That's continuous (i.e. in $C(M)$), but unfortunately not bounded. To get around this problem we fix a point $x_0 \in M$ and define, for $x \in M$:

$$f_x = d_x - d_{x_0}.$$

The metric in $B(M)$ corresponding to uniform convergence is:

$$D(f,g) = \sup_{z \in M} |f(z) - g(z)|.$$

Problem 1. (i) Show that $f_x \in B(M)$, for each $x \in M$. (That is, for each fixed $x \in M$ we may find a constant $C(x) > 0$ so that $|f_x(y)| \leq C(x)$, for all $y \in M$.)

(ii) Show that, for all $x, y \in M$:

$$D(f_x, f_y) \leq d(x, y).$$

(iii) Show that, for all $x, y \in M$:

$$D(f_x, f_y) \geq d(x, y).$$

(*Hint:* Definition of *sup*: set $z = y$).

We conclude the assignment $x \mapsto f_x$ defines an *isometric embedding* Φ from (M, d) to the complete metric space $(B(M), D)$; indeed Φ maps M to the subspace $CB(M)$ of continuous bounded functions on M (which, being closed in $B(M)$ under uniform convergence, is also complete for the same metric D, i.e. a Banach space.)

The image $\Phi(M)$ will almost never be dense in $CB(M)$, so the last step is to define:

$$\hat{M} = \overline{\Phi(M)}$$

(closure in $CB(M)$, with respect to the metric D.) We now have a pair $(\Phi, \hat{M})$ satisfying all the conditions for the completion: $(\hat{M}, D)$ is complete (as a closed subset of a complete metric space), $\Phi : (M.d) \to (\hat{M}, D)$ is an isometric embedding, and $\Phi(M)$ is dense in $\hat{M}$.

3. Uniqueness of Metric Completion.

Definition. A *metric completion* of a metric space (M, d) is a pair $(f, \hat{M})$, where $(\hat{M}, D)$ is a complete metric space and $f : M \to \hat{M}$ is an isometric embedding (in particular, injective), with $f(M)$ dense in $\hat{M}$.

Having shown metric completions exist, we proceed to uniqueness. They can't be strictly unique, since $(g \circ f, \tilde{M})$ will also be a metric completion of (M,d), if $g : \hat{M} \to \tilde{M}$ is an isometry to a second complete metric space $(\tilde{M}, \tilde{D})$. But this is the only indeterminacy.

Theorem. Let $(f, \hat{M})$ and $(\tilde{f}, \tilde{M})$ be metric completions of a metric space (M,d), with metrics $D, \tilde{D}$ (resp.) Then there exists an isometry (in fact a unique one) $g : \hat{M} \to \tilde{M}$ so that:

$$\tilde{f} = g \circ f.$$

Proof. First define g on $f(M) \subset \hat{M}$ via:

$$g(y) = \tilde{f}(x), \text{ where } f(x) = y$$

(there is a unique such x, since $y \in f(M)$ and f is injective.) Clearly $g \circ f = \tilde{f}$ holds on M. Note that g is isometric on $f(M)$, since:

$$\tilde{D}(g(y), g(y')) = \tilde{D}(\tilde{f}(x), \tilde{f}(x')) = d(x,x') = D(y,y') \text{ with } f(x) = y, f(x') = y'.$$

In particular, g is *uniformly continuous* on $f(M)$, and thus (since $\hat{M}$ is complete) extends continuously to the closure of $f(M)$, which is all of $\hat{M}$. And it is easy to see that the extension (still denoted by g, now defined on $\hat{M}$) is an isometric embedding from $\hat{M}$ to $\tilde{M}$ (in particular injective.)

Problem 2. Show that g is surjective, that is: $g(\hat{M}) = \tilde{M}$.

Hint. Let $z \in \tilde{M}$, and consider a sequence $x_n \in M$ so that $\tilde{f}(x_n) \to z$ (which exists since $\tilde{f}(M)$ is dense in $\tilde{M}$). Show that the limit $\lim f(x_n) = y$ exists, and that $g(y) = z$.

Problem 3. Let (M,d) be the real line with the metric:

$$d(s,t) = \frac{|s-t|}{\sqrt{1+s^2}\sqrt{1+t^2}}.$$

(i) Show that (M,d) is not complete;

(ii) Identify the completion of (M,d).

Hint: Consider the map $h : M \to R^2$:

$$h(s) = (\frac{s}{1+s^2}, \frac{s^2}{1+s^2}).$$

Show that h is an isometric embedding from (M,d) to R^2 (with the euclidean metric), with image the circle:

$$x^2 + (y - \frac{1}{2})^2 = \frac{1}{4},$$

except for the point $(0,1)$.

4. Completion of normed linear spaces. Let $(E, ||\cdot||)$ be a normed linear space. A *completion* of E is a pair (L, F), where (F, p) is a complete normed linear space (i.e. a *Banach space*) and $L : E \to F$ is a linear isometric embedding:

$$p(Lv) = ||v||, \text{ for all } v \in E$$

with dense image: $L(E)$ is dense in F (with respect to the norm p.)

Existence of the completion can be shown just as for metric spaces: introduce in the set $C(E)$ of Cauchy sequences on E the equivalence relation:

$$(x_n) \equiv (y_n) \text{ if } ||x_n - y_n|| \to 0.$$

Then define F as the set of equivalence classes. It is easy to introduce a vector space structure on F, via:

$$V + W = [(z_n)], \quad z_n = x_n + y_n, \quad V = [(x_n)], \quad W = [(y_n)].$$

$$\lambda V = [\lambda x_n], \quad \lambda \in R, \quad V = [(x_n)].$$

(It is easily shown this is independent of the choices of Cauchy sequences representing V, W.) Then define a norm on F via:

$$p(V) = \lim ||x_n|| \text{ if } V = [(x_n)].$$

It is straightforward to show the limit exists, $p(V)$ is well-defined (independent of the sequence chosen to represent V) and defines a norm on F, and that this norm is complete (Cauchy sequences converge.) The embedding L of E into F via equivalence classes of constant sequences is evidently a linear isometric embedding.

Problem 4. Show that $L(E)$ is dense in F (with respect to the norm p.)

Uniqueness of the norm completion is the statement: if $T : E \to H$ is a linear isometric embedding into a second Banach space (H, q) with dense image, then there exists an isometry:

$$U : F \to H, \quad q(UV) = p(V) \text{ for all } V \in F, \quad U \text{ a linear isomorphism},$$

satisfying: $U \circ L = T$. This is proved just as for metric spaces.