NORMAL SPACES, REGULAR SPACES, URYSOHN METRIZATION.

Definition. A topological space X is *normal* if it is Hausdorff and open sets separate disjoint closed sets:

$$A, B \subset X \text{ closed, disjoint } \Rightarrow \exists U \supset A, V \supset B; U, V \subset X \text{ open, disjoint.}$$

This condition is equivalent to:

$$E \subset X \text{ closed }, V \supset E \text{ open } \Rightarrow \exists U \subset X \text{open}, E \subset U \subset \overline{U} \subset V.$$

The main reason to be interested in normal spaces is the following theorem.

Urysohn metrization theorem. Let X be Hausdorff and second-countable. Then X is metrizable if and only if X is normal.

The following lemma used in the proof is also important:

Urysohn's lemma. Let X be normal, $E, F \subset X$ closed and disjoint. Then there exists $f \in C(X, [0,1])$ so that $f \equiv 0$ on E and $f \equiv 1$ on F.

Conversely, if X is Hausdorff and for each $E, F \subset X$ closed and disjoint we may find such an f, then X is normal (this part is easy.)

In other words, for a Hausdorff space X, the conditions "open sets separate disjoint closed sets' and 'continuous functions separate disjoint closed sets' are *equivalent.*

First the good news:

N1) Every metric space is normal. The easiest way to see this is to realize the function

$$f(x) = \frac{d(x, E)}{d(x, E) + d(x, F)} \quad E, F \subset X \text{closed, disjoint}$$

solves the problem in Urysohn's lemma.

N2) Every compact Hausdorff space is normal. This has a two-step proof. First we go from Hausdorff to regular (definition below), then from regular to normal, in each case by taking finite subcovers of open covers.

N3) *Tietze extension theorem.* Let X be normal, $E \subset X$ closed $f : E \to R$ continuous and bounded (say $|f| \leq M$ on E.) Then we may extend f to $\bar{f} \in C(X)$ continuous, with the same bound $|\bar{f}| \leq M$ (that is, $f = \bar{f}$ at points of E.)

(And conversely, if such an extension exists for each f and E, then X is normal, assuming Hausdorff.)

Unfortunately normal spaces have two undesirable properties:

N4) A subspace $Y \subset X$ of a normal space is not necessarily normal. However:

Exercise 1. A *closed* subspace $Y \subset X$ of a normal space X is also normal (with the induced topology.) (Prove and use the fact that if E is closed in Y, then it is also closed in X (since Y is closed in X.))

N5) The product $X = X_1 \times X_2$ of two normal spaces is not necessarily normal (for the product topology.)

One could live with (N4), but from the point of view of the metrization theorem (N5) is really a pity: the hypotheses 'Hausdorff' and 'second countable' persist under taking subspaces or (finite) products, as does the conclusion 'metrizable'–but the condition 'normal' doesn't! Not a satisfactory state of affairs.

Exercise 2. Prove that if X_1, X_2 are second countable, then so is $X_1 \times X_2$ (with the product topology.)

To remedy this state of affairs, we consider a weaker condition.

Definition. A topological space X is *regular* if it is Hausdorff and 'open sets separate points from closed sets':

$$(R) \quad \forall F \subset X \text{ closed } \forall x \notin F, \exists U_x, V_x \text{ open, disjoint so that } x \in U_x, V_x \supset F.$$

The following are equivalent ways to state the definition:

R1) For each $x \in X$, each V_x open neighborhood of x, we may find a second open neighborhood U_x of x so that $U_x \subset \overline{U_x} \subset V_x$.

(We say 'the topology admits a local basis of closed sets'.) To see this is equivalent to (R), set $F = (V_x)^c$ (closed).

R2) $\forall x \in X, F \subset X$ closed $, x \notin F, \exists U_x$ neighborhood of $x, \overline{U_x} \cap F = \emptyset$. (Set $V_x = F^c$ (open) in (R1)).

Properties. Regular spaces do not suffer from the same problems as normal spaces:

R3) Any subspace $Y \subset X$ of a regular space is regular (**Exercise 3.**)

R4) The product $X = X_1 \times X_2$ of two regular spaces is regular (and if X is regular, so are X_1 and X_2.) **(in class)**

R5) Compact Hausdorff spaces and metrizable spaces are regular. (Clearly, since normal spaces are regular.)

The statement analogous to Urysohn's lemma fails for general regular spaces: continuous functions *do not* necessarily separate points and closed sets, in a general regular space. (Nothing is perfect!)

However, as far as the metrization theorem goes, we need not be concerned.

R6) *Proposition.* Let X be a second countable space. If X is regular, then X is normal.

(Thus 'normal', 'regular' and 'metrizable' are all equivalent, for second countable spaces.)

Proof. (outline.) Let E, F be closed, disjoint subsets of X. Using Lindelöf's theorem and the definition (R) of 'regular', we find countable open covers of E and F:

$$E \subset \bigcup_{n\geq 1} U_n, \quad F \subset \bigcup_{m\geq 1} V_m, \quad U_n \cap V_n = \emptyset \forall n \geq 1.$$

Then set:

$$A_n = U_n \setminus (\overline{V_1} \cup \ldots \overline{V_n}), \quad B_n = V_n \setminus (\overline{U_1} \cup \ldots \overline{U_n}).$$

It is easy to check we still have the open covers:

$$E \subset A := \bigcup_{n\geq 1} A_n, \quad F \subset B := \bigcup_{m\geq 1} B_m.$$

It is easy to show that $A_n \cap B_m = \emptyset \quad \forall m, n \geq 1$. Thus A, B are disjoint open neighborhoods of E, F, and X is regular.

The following is another nice property.

Definition. A Hausdorff topological space X is *locally compact* if it admits local bases of compact neighborhoods, that is:

For each $x \in X$, each V_x open neighborhood of x, we may find a second open neighborhood U_x of x so that $U_x \subset \overline{U_x} \subset V_x$. and $\overline{U_x}$ is a compact subset of X.

Proposition. Any locally compact Hausdorff space X is regular.

Remark 1. A locally compact Hausdorff space is not always normal, though.

Remark 2. Thus we see that *a second countable, locally compact Hausdorff space is metrizable*, an important consequence.

Remark 3. The proposition is plausible, since 'regular' (unlike 'normal' or 'Hausdorff') is a *local property:* if each point of X has an open neighborhood which, regarded as a space in its own right via the induced topology, is regular, then X itself is regular (using definition R2).

Example. A topological space X is an *n-dimensional topological manifold* if it is Hausdorff, second countable, and locally homeomorphic to R^n. (That is, each $x \in X$ has a neighborhood homeomorphic to an open ball in R^n.) So any n-manifold is metrizable.

Proof of Proposition. Let $x \in X$. Let U_x be an open neighborhood of x. Using definition R2, we want to find a second open neighborhood B of x so that $\overline{B} \subset U_x$.

Since X is locally compact, we may find V (open nbd. of x) with $\overline{V}$ compact. From the definition of open set, we may find W_x open in X (containing x) so that $W_x \subset U_x \cap V$. In particular, W_x is open in V (and hence in the compact space $\overline{V}$).

Now, compact spaces are regular, so using definition R2) we find $B \subset \overline{V}$ open in $\overline{V}$, containing x, and with $\overline{B} \subset W_x$. (And now comes the subtle part.) Since $W_x \subset V$, also $B \subset V$, so B is open in V (not just in $\overline{V}$), and since V is open in X, B is open in X too. And $\overline{B}$ is contained in W_x, hence also in U_x. Thus B satisfies the conditions required by R3).

Topological embeddings. A continuous map $f : X \to Y$ of topological spaces is an *embedding* if it is injective, and defines a homeomorphism from X to $f(X)$ (as a topological space with the topology induced form Y.)

Equivalently, f is continuous, injective, and *open*: maps open subsets of X to open subsets of $f(X)$ (intersections of open sets of Y with $f(X)$.)

The *idea of proof* of the Urysohn metrization theorem is to use the existence of a countable basis and normality to define a continuous map from X to a metric space C (the Hilbert cube), and show this map is an embedding. Then $f(X)$ inherits the metric from C, so X is metrizable.

The Hilbert cube.

We first consider the space l^2 of square-summable series of real numbers:

$$l^2 = \{x = (x_i)_{i \geq 1}; \sum_{i=1}^{\infty} x_i^2 < \infty\}.$$

Proposition. l^2 is a separable Banach space. (shown in class.)

We define the Hilbert cube as the subset of l^2:

$$C = \{x; (\forall i \geq 1)|x_i| \leq \frac{1}{i}\}.$$

Notation In the following, we write x^i (superscript) for the i^{th} component of $x \in l^2$; so $x^i \in [-\frac{1}{i}, \frac{1}{i}]$ if $x \in C$.

Proposition. Let $(x_n)_{n\geq 1}$ be a sequence in C, and let $x_0 \in l^2$. Then $x_n \to x_0$ in l^2 norm if and only if $x_n^i \to x_0^i$, for each $i \geq 1$.

Proof. (i) Assume $x_n^j \to x_0^j$, for each $j \geq 1$. (In particular $x_0 \in C$.) Given $\epsilon > 0$, choose $N \geq 1$ so that $\sum_{j\geq N+1}(1/j^2) < \epsilon^2$.

For each $1 \leq j \leq N$, pick $N_j \geq N$ so that $|x_n^j - x_0^j|^2 \leq \frac{\epsilon^2}{N}$ if $N \geq N_j$. Then let $M = \max\{N_j; 1 \leq j \leq N\}$. For $n \geq N$:

$$||x_n - x_0||^2 = \sum_{i=1}^{N} |x_n^i - x_0^i|^2 + \sum_{i=N+1}^{\infty} |x_n^i - x_0^i|^2$$

$$\leq N\frac{\epsilon^2}{N} + 2\sum_{i=N+1}^{\infty} (x_n^i)^2 + (x_0^i)^2 \leq \epsilon^2 + 4\sum_{j\geq N+1} \frac{1}{j^2} < 5\epsilon^2.$$

(Note that this is false for general sequences in l^2: convergence of each component *does not* imply convergence in L^2 norm!)

(ii) The converse is clear: if $x_n \to x_0$ in l^2 norm, for each $i \geq 1$:

$$|x_n^i - x_0^i| \leq ||x_n - x_0|| \to 0.$$

Corollary. In particular, we see that C is closed in l^2, hence is a complete metric space.

We also see that the l^2 norm induces in C the product topology:

$$C = \Pi_{i=1}^{\infty}[-\frac{1}{i}, \frac{1}{i}].$$

Proposition. C is compact.

Proof (outline). It is enough to show C is totally bounded. Given $\epsilon > 0$, choose $N \geq 1$ so that $\sum_{i\geq N+1}(1/i^2) < \epsilon^2/2$.

For each $1 \leq i \leq N$, let F_i be a $\frac{\epsilon}{\sqrt{2N}}$-net in $[-\frac{1}{i}, \frac{1}{i}]$. Then define:

$$F = \{x; x^i \in F_i \text{ for } 1 \leq i \leq N; x^i = 0, i \geq N+1\}.$$

F is a finite set, and it is easy to check that F is an ϵ-net for C (that is, for any $y \in C$ we may find $x \in F$ so that $||x-y||^2 < \epsilon^2$): for each $1 \leq i \leq N$, find $x^i \in F_i$ so that $|x^i - y^i| < \epsilon/\sqrt{2N}$. Then set the remaining x^i equal to 0.

Proof of the Urysohn metrization theorem. Let $\mathcal{B} = (B_n)_{n\geq 1}$ be a countable basis of X.

Since X is regular, for each $x \in X$ we may find n and m so that:

$$x \in B_m \subset \overline{B_m} \subset B_n.$$

Let's call a pair (B_m, B_n) of open sets in $\mathcal{B}$ *admissible* if $\overline{B_m} \subset B_n$. Denote by $\mathcal{P}$ the set of admissible pairs, which is countable (being a subset of $\mathcal{B}\times\mathcal{B}$). So we take an enumeration $\mathcal{P} = (P_i)_{i\geq 1}$, $P_i = (B_{m_i}, B_{n_i})$.

By Urysohn's lemma, we may find for each $i \geq 1$ a continuous function $f_i : X \to [-1/i, 1/i]$ so that $f_i \equiv 1/i$ on B_{m_i}, $f_i \equiv 0$ on $B_{n_i}^c$.

Define $f : X \to C$ via $f(x)^i = f_i(x)$, for each $i \geq 1$. We *claim* this f is an embedding of X into C.

1. f is continuous. Since X is first countable, it is enough to consider $x_n \to x_0$ in C. Then $f_i(x_n) \to f_i(x_0)$ for each $i \geq 1$ (since f_i is continuous); hence, as seen above, $f(x_n) \to f(x_0)$ in C.

2. f is injective. Let $x \neq y$ be points of C. Then we may find (since X is Hausdorff) $B_n \in \mathcal{B}$ so that $x \in B_n, y \in B_n^c$. Since X is regular, we may also find $B_m \in \mathcal{B}$ so that $x \in B_m \subset \overline{B_m} \subset B_n$. Thus the pair (B_m, B_n) is admissible, equal to $P_i = (B_{m_i}, B_{n_i})$ for some $i \geq 1$. We see that $f_i(x) = 1/i, f_i(y) = 0$, so $f(x) \neq f(y)$.

3. f maps open sets of X to open subsets of $f(X)$ (in the topology induced from C).

It is enough to show that for each $B_k \in \mathcal{B}$, $f(B_k) = A_k \cap f(X)$, for some open set $A_k \subset C$.

Fix $k \geq 1$, and let $J_k \subset N$ be the set of $n \geq 1$ so that (B_n, B_k) is an admissible pair.

For each $i \geq 1$, the set $C_i = \{y \in C; y^i > 0\}$ is open in C (why?) Thus the set

$$A_k = \bigcup_{i\in J_k} C_i$$

is also open in C. We *claim* $f(B_k) = A_k \cap f(X)$.

First, let $x \in B_k$. Then $\exists m \geq 1$ with $x \in B_m \subset \overline{B_m} \subset B_k$ (since X is regular) , so $(B_m, B_k) = P_i$ (for some i) is admissible, with $i \in J_k$.

Thus $f_i(x) = 1/i > 0$, and $f(x) \in C_i$, so $f(x) \in A_k$. This shows $f(B_k) \subset A_k \cap f(X)$.

Conversely, if $f(x) \in A_k$, then $f_i(x) > 0$ for some $i \in J_k$. Since $f_i \equiv 0$ on B_k^c, this implies $x \in B_k$. So $A_k \cap f(X) \subset f(B_k)$, concluding the proof.

Exercise 4. Define 'locally metrizable space'. Prove that a Hausdorff, separable, locally metrizable space X is metrizable.