COMPACTNESS AND LOCAL COMPACTNESS (all spaces assumed Hausdorff.)

Definition. X is *countably compact* if any countable open cover admits a finite subcover.

Thus any compact space is countably compact, and on a Lindelöf space the concepts are equivalent (for example on any 2nd countable space.)

Def. X is *sequentially compact* if any sequence on X admits a convergent subsequence.

Def. A point $z \in X$ is an *accumulation point* of a sequence (x_n) in X if any neighborhood of X contains infinitely many points of the sequence.

1. If X is first-countable, $z \in X$ is an accumulation point of (x_n) iff some subsequence of (x_n) converges to z.

2. If X is countably compact, any sequence in X has an accumulation point. As a consequence, if X is countably compact (in particular, if X is compact) and first-countable, X is sequentially compact.

Hint: If (x_n) is a sequence without any accumulation points, each x_n has a neighborhood U_n with no other points of the sequence. Let $A = \{x_1, x_2, \ldots\}$, a closed subset of X (why?). Adding A^c to the U_n gives an open covering of X, which has no finite subcover.

3. If X is sequentially compact, X is countably compact. (Hence for first-countable spaces, these concepts are equivalent.)

Hint. Let $\{U_n\}_{n\geq 1}$ be a countable open cover of X. If it has no finite subcover, we may define a sequence (x_n) in X taking:

$$x_1 \in X \setminus U_1, x_2 \in X \setminus (U_1 \cup U_2), \ldots, \quad x_n \in X \setminus (\bigcup_{i=1}^{n} U_i).$$

Note $x_n \notin U_i$ if $n \geq i$: each U_i has only finitely many sequence elements. But if $z \in X$, z is in some U_{n_0}.

4. The space X of all functions from $[0,1]$ to itself (with the topology of pointwise convergence) is compact (by Tychonoff's theorem), but not first-countable. And indeed it is not sequentially compact: exhibit a sequence $f_n \in X$ with no convergent subsequence.

Hint. Let f_n be the function (with image in $\{0,1\}$) that assigns to each $x \in [0,1]$ the nth. digit in its base 2 expansion (terminating in 0s if x is a dyadic rational). For any increasing sequence $(n_k)_{k\geq 1}$, Let $x_0 \in [0,1]$ have the binary expansion $0.a_1a_2.a_3...$: $a_k = 0$ if k is even, $a_k = 1$ if k is odd. Then $f_{n_k}(x_0)$ does not converge.

5. If X is second countable and sequentially compact, X is compact.
Hint. X is first countable (hence countably compact) and Lindelöf.

6. If X is first countable and countably compact, X is regular. (It follows via Urysohn metrization that compact second countable spaces are metrizable!)
Hint. Since X is Hausdorff, given $x \in X$ we may find a countable local basis (V_n) at x which decreases to x:

$$V_1 \supset V_2 \supset V_3 \supset \dots, \qquad \bigcap_{n \geq 1} V_n = \{x\}.$$

Let V be a nbd of x. Adding V to $\{(\overline{V_n})^c; n \geq 1\}$ one gets a countable cover of X. Taking a finite subcover we find V_N so that $\overline{V_N} \subset V$, showing X is regular.

COMPACTNESS IN METRIC SPACES

7. (X, d) compact metric is sequentially compact; in particular, X is complete.
Hint. This follows from the fact X is first countable (and countably compact). For a direct proof, if a sequence with not convergent subsequence exists, then for each $x \in X$ there is an open ball $B(x, r_x)$ including only finitely many sequence elements; taking a finite subcover leads to a contradiction. (This uses first countability too.)

Def. (X, d) compact metric is totally bounded: for each $R > 0$, finitely many balls of radius R cover X.

8. (i) In a totally bounded metric space, any sequence has a Cauchy subsequence (nested balls argument.)
(ii) If (X, d) is totally bounded and complete, X is sequentially compact.

9. A totally bounded metric space is separable (hence second-countable.)

10. If (X, d) is a sequentially compact metric space, X is complete and totally bounded.
Hint. If not, one may find $R > 0$ and $x_1 \in X, x_2 \notin B(x_1, R), \dots x_n\ / inB(x_1, R) \cup \dots B(x_{n-1}, R)$, so $d(x_{n+1}, x_i) \geq R$ for $i = 1, \dots, x_n$: this sequence has no convergent subsequence.

So far we see that a compact metric space is sequentially compact; and that a metric space is sequentially compact iff it is complete and totally bounded. To close the circle, we need:

10. *Lebesgue number lemma.* Any open cover $\mathcal{F}$ of a sequentially compact metric space has a *Lebesgue number*: $L > 0$ so that any subset $C \subset X$ with diameter less than or equal to L is contained in a set U of the cover.

Hint: If not, we may find a sequence of closed sets C_n with diameter less than $1/n$, not contained in any open set in $\mathcal{F}$, and $x_n \in C_n$. Then a subsequence $x_{n_i} \to z \in U$, $U \in \mathcal{F}$ open. But also $B(z, \epsilon) \subset U$ for some $\epsilon > 0$, and for i large, $C_{n_i} \subset U$ (show this). Contradiction.

11. Any sequentially compact metric space (X, d) is compact.

Hint. Use the fact X is totally bounded: given an arbitrary open cover, consider its Lebesgue number L, and cover X by finitely many balls of radius $L/3$.

CONCLUSION: For a metric space, 'compact', 'sequentially compact', 'countably compact' and 'complete and totally bounded' are all equivalent. Compact metric spaces are separable and second countable; and any open cover admits a Lebesgue number.

12. If X, Y are separable metric spaces with X *compact*, then $C(X; Y)$ is separable metric (with the topology of uniform convergence.) If X is not compact, this is false in general. [Solved in lecture.]

Def. A metric space (X, d) is *locally separable* if for each $x \in X$ there exists an open ball $B(x, r_x)$ containing a countable dense subset.

13. (i) If (X, d) is connected and locally separable, then X is separable. [Solved in lecture]

(ii) If (X, d) is connected and locally compact, then X is separable (and second countable.)

14. A metric space X is separable if and only if it is homeomorphic to a subset of a compact metric space.

LOCALLY COMPACT SPACES (Def: each point $x \in X$ has a precompact open neighborhood.

Already discussed: (i) Locally compact Hausdorff spaces are regular (even completely regular); (ii) Locally compact Hausdorff spaces are Baire spaces; (iii) Connected, locally compact metrizable spaces are separable.

15. X (Hausdorff) is loc. compact $\Leftrightarrow$ each $x \in X$ admits a local basis of precompact neighborhoods.

Hint. ETS any open nbd U of X contains a precompact one. Let V be a precompact neighborhood of x. The open nbd of x $A = U \cap V$ contains an open nbd B s.t. $x \in B \subset \overline{B} \subset A$ (why?) Now B is precompact, contained in U.

Def. $S \subset X$ is *locally closed* if any $x \in S$ has a neighborhood $U \subset X$ (open), so that $U \cap S$ is *closed* in U.

Proposition 1. S is loc. closed $\Leftrightarrow S = A \cap F$, where A is open in X, F is closed in X [Proved in lecture]

16. Example: consider the graph $G \subset R^2$ of

$$f : (0, \infty) \to R, \qquad f(x) = \sin \frac{1}{x}.$$

(i) G is a closed subset of $(0, \infty) \times R$, but not of R^2.

(ii) G is locally closed in R^2; find sets A and F as in proposition 1.

Proposition 2. X loc. compact Hausdorff, $S \subset X$ loc. closed $\Rightarrow S$ is loc. compact.

Proof. By prop. 1, $S = U \cap F$, with U open, F closed in X. Given $x \in S$, there is a precompact nbd V of x with $\overline{V} \subset U$. Then $V \cap S$ is a nbd of x in S, with closure (in the induced topology) $\overline{V} \cap S$. But:

$$\overline{V} \cap S = \overline{V} \cap (U \cap F) = \overline{V} \cap F,$$

which is compact. Thus any $x \in S$ has a precompact open nbd. (in S with the induced topology.)

17. Let X be locally compact Hausdorff.

(i) Closed subsets of X are locally compact.

(ii) Open subsets of X are locally compact.

Proposition 3. X loc. compact Hausdorff, $S \subset X$ dense in X and locally compact $\Rightarrow S$ is *open* in X. [Proof seen in Lecture.]

18. (i) Any locally compact metric space is open in its completion.

(ii) Any locally compact metric space is *completely metrizable* (i.e. admits a complete metric defining the same topology.)

A locally compact Hausdorff space X admits an *Alexandrov compactification* X^*, meaning a compact Hausdorff space and an embedding $\varphi : X \to X^*$ with $X^* \setminus \varphi(X) = \{\omega\}$, the 'point at infinity'. Neighborhoods of ω in X^*

have the form $\{\omega\} \sqcup U$ (disjoint union), where $U \subset X$ is the complement of a compact subset of X. [Proof given in lecture.]

Such a compactification is unique: if $\psi : X \to \tilde{X}$ is a second one, $\tilde{X} = X \sqcup \{\tilde{\omega}\}$, one may find a homeomorphism $h : X^* \to \tilde{X}$, $h(\omega) = \tilde{\omega}$, so that $\varphi = \psi \circ h_{|X}$.

18.5 Use the existence of the Alexandrov compactification to prove that locally compact Hausdorff spaces are completely regular.

Def. The Alexandrov compactification of a loc. cpt Hausdorff space X is *countable at infinity* if the point at infinity has a countable basis of neighborhoods.

19. This happens iff X is *σ-compact*: $X = \bigcup_{i \geq 1} K_i$, $K_i \subset X$ compact, which may be assumed to be increasing, $K_i \subset int(K_{i+1})$.

Extension of continuous functions and maps.

Def. Let X be loc. compact Hausdorff and non-compact, Y be a Hausdorff space, $f : X \to Y$ continuous, $y_0 \in Y$. We say $\lim_{x\to\infty} f(x) = y_0$ if for any neighborhood V of y_0 in Y, we may find a compact set $K \subset X$ so that:

$$x \in X \setminus K \Rightarrow f(x) \in V.$$

20. $f : X \to Y$ extends continuously to the Alexandrov compactification X^* (via $F : X^* \to Y, F(\omega) = y_0$) iff $\lim_{x\to\infty} f(x) = y_0$.

Def. Let X, Y be both locally compact Hausdorff. A continuous map $f : X \to Y$ is *proper* if the preimage of any compact set is compact.

21. f extends to a continuous map $F : X^* \to Y^*$ of the Alexandrov compactifications (with $F(\omega_X) = \omega_Y$) iff f is proper.

Remark on problem 21. A related concept is that of *perfect map* [Munkres, p.199]: A continuous surjective map $f : X \to Y$ is *perfect* if it is closed and all level sets $f^{-1}(y)$ are compact.

This implies the properties (i) Hausdorff; (ii) regular; (iii) locally compact; (iv) second countable are inherited by Y, if satisfied by X.

22. Let $f : X \to Y$ be continuous, surjective and closed.

(i) For each $y \in Y$ and any $U \subset X$ open neighborhood of the preimage (level set) $f^{-1}(y)$, there exists $V \subset Y$ open neighborhood of y so that $f^{-1}(V) \subset U$. (In fact this 'continuity of level sets' characterizes closed maps.)

(ii) Let $y \in Y$, let $U \subset X$ be an open neighborhood of $f^{-1}(y)$. Then $f(U)$ contains an open neighborhood $V \subset Y$ of y.

23. Perfect maps are proper. *Hint:* Problem 22(i).

Remark. The converse is true, under the hypothesis the topologies of X and Y are *compactly generated* ([Munkres p. 283]): a subset $A \subset X$ is open in X if $A \cap C$ is open in C, for each $C \subset X$ compact subspace.

As proved in [Munkres, p.283]: locally compact spaces and first countable spaces (in particular, metric spaces) are compactly generated.