PROPER (or HEINE-BOREL) METRIC SPACES

Def. (X, d) is *proper* (or HB) if closed bounded sets are compact.

Ex.1 $\mathbb{R}^n$ is HB. (Since bounded sets are totally bounded.)

Ex.2 For X compact metric, $C(X)$ (with the sup norm) is complete separable metric, but not HB.

Exercise 1. Let (X, d) be a proper metric space. Then X is complete, locally compact and σ-compact.

The converse is not true: it is easy to destroy the HB property.

Ex. 3. Let (X, d) be a non-compact metric space. Then the metric $d_{min} = \min\{d, 1\}$ is equivalent to d (same topology), locally coincides with d (in a neighborhood of the diagonal in $X \times X$), but is not HB. In particular, (X, d_{min}) is locally compact, σ-compact and complete, if d is.

Exercise 2. A metric space (X, d) is proper if and only if the distance function to a point $x \mapsto d(x, x_0)$ is a proper function on X (preimage of compact is compact.) Hence the name.

In the paper [Williamson-Janos], the following theorems are proved:

Theorem 1. If (X, d) is locally compact, σ-compact, there exists an equivalent metric (same topology) which is HB.

Idea of proof. Let (K_n) be a compact exhaustion of X, $K_n \subset int(K_{n+1})$. Let $f_n : X \to [0, 1]$ continuous satisfy: $f_n \equiv 0$ on K_n, $f_n \equiv 1$ on K^c_{n+1}. Consider the metric on X:

$$d'(x, y) = d(x, y) + \sum_{n \geq 1} |f_n(x) - f_n(y)|.$$

Exercise 3. Show that d and d' are equivalent on X. (It suffices to show that $\lim x_n = x$ with respect to d iff this holds with respect to d'.)

Note that any bounded set must be contained in some K_n, so (X, d') is HB. In particular complete, even if (X, d) isn't. This gives a proof that locally compact, σ-compact metric spaces are completely metrizable.

Theorem 2. Let (X, d) be locally compact, σ-compact and complete. Then X admits a HB metric that locally coincides with d.

Idea of proof. Let $\mathcal{W}$ be an open cover of X. Define a new metric on X as follows. Given $x, y \in X$, find a chain of points from x to y:

$$x_0 = x, x_1, \ldots, x_N = y, \quad \text{and } \forall i = 0, \ldots, N-1 : x_i, x_{i+1} \in A, \text{ for some } A \in \mathcal{W}.$$

Then let $d_{\mathcal{W}}(x, y)$ be the infimum of $\sum_{i=0}^{N-1} d(x_i, x_{i+1})$ over all such chains from x to y. This '$\mathcal{W}$ distance' is a metric locally equivalent to d.

To prove the theorem, the authors consider a compact exhaustion (K_n) of X, and the associated open cover by the sets $int(K_{n+1}) \setminus K_n$ and show (using completeness) that the metric associated with this cover is HB.

Gromov's Hopf-Rinow Theorem.

Def. Let $\gamma : [0, 1] \to X$ be a continuous curve, where (X, d) is metric. The length of γ is defined as:

$$L[\gamma] = \sup\{\sum_i d(\gamma(t_i), \gamma(t_{i+1}))\},$$

where the supremum is taken over all partitions of $[0, 1]$. (In general, this will be infinite–for example, for a non-rectifiable curve in R^n.)

Def. (X, d) (path connected) is a *length space* if $d(x, y)$ is the inf of the lengths of curves joining x to y.

Def. A path $\gamma : [0, L] \to X$ is a *geodesic segment* if it is an isometry from $[0, L]$ (with its usual metric) to X. X is a *geodesic length space* if any two points can be joined by a geodesic segment. (Example: $R^2 \setminus \{0\}$ with the metric induced from R^2 is a length space, but not a geodesic one.)

On a Riemannian manifold M, lengths of C^1 ucrves are defined by integrating the lengths of their tangent vectors, and Riemannian distance d is defined by taking the inf of the lengths of C^1 curves joining two points. The *Hopf-Rinow theorem* says the following conditions are equivalent: (i) (M, d) is a complete metric space; (ii) (M, d) is HB. And if either condition holds, (M, d) is a geodesic length space.

Gromov's version of this for general metric spaces is:
Theorem. Any complete, locally compact length space is HB (and geodesic.)

Idea of proof. Fix $x_0 \in X$ and let $J \subset \mathbb{R}_+$ be the set of $r > 0$ such that:

$$K(x_0, r) := \{x \in X; d(x_0, x) \leq r\}$$

is compact. From local compactness, J is non-empty and open; either $J = \mathbb{R}_+$ (in which case X is proper) or $J = (0, a)$ for some $a > 0$.

Observe the following property of length spaces: if $d(x, z) < a + b$, we may find $y \in X$ so that $d(x, y) < a, d(y, z) < b$. Indeed just find a path from x to z with length less than $a + b$, and pick y suitably along that path.

So by contradiction assume $J = (0, a)$ and let (x_n) be a sequence in $K(x_0, a)$. Using this observation, for each $n \geq 1, k \geq 1$ find $y_{n,k}$ so that:

$$y_{n,k} \in K(x_0, a - \frac{1}{k}), \quad d(y_{n,k}, x_n) < \frac{2}{k}.$$

For each k fixed, $(y_{n,k})_n$ has a convergent subsequence. Then use a diagonal argument to find n_j so that $(y_{n_j,k})_j$ is convergent for each k. Considering:

$$d(x_{n_j}, x_{n_l}) \leq d(x_{n_j}, y_{n_j,k}) + d(y_{n_j,k}, y_{n_l,k}) + d(y_{n_l,k}, x_{n_l}),$$

where the middle term is small for j and l large and the sum of the other two is less than $\frac{4}{k}$, taking k large enough we see that (x_{n_j}) is a Cauchy sequence, hence convergent since X is complete. Thus any sequence in $K(x_0, a)$ has a convergent subsequence, so this set is compact. Contradiction.

(Reference: John Roe, *Lectures on Coarse Geometry.*)

We conclude with a proposition involving this circle of ideas:

Proposition. Let M be a locally compact metric space. The following conditions are equivalent:

1- M has a countable basis;
2- M is σ-compact;
3-M admits an equivalent Heine-Borel metric;
4- The Alexandroff compactification $M^* = M \sqcup \{\omega\}$ is metrizable.

Proof. (1)$\Rightarrow$ (2): Each $x \in M$ is in an open, precompact set U_x. Since M is Lindelöf, the open cover $\{U_x\}_{x \in M}$ admits a countable subcover $\{U_n\}_{n \geq 1}$. Thus M is contained in the countable union of compact sets $\{\overline{U}_n\}_{n \geq 1}$.

(2) $\Rightarrow$ (3): this is Theorem 1 above.

(3) $\Rightarrow$ (2): Exercise 1 above.

(2)$\Rightarrow$ M^* has a countable basis: It is clear that M does (that is, (2) $\Rightarrow$ (1)), and also that ω has a countable local basis. The union of these two bases is a countable basis for the topology of M^* (informally speaking; fill in the details as an exercise.) Since M^* is normal (being compact), metrizability (4) follows from Urysohn metrization.

Thus (1),(2),(3) are equivalent, and each implies (4). But (4)$\Rightarrow$ (1) is clear, since compact metrizable spaces are second-countable, and this is inherited by subspaces.