Stone - Weierstrass theorem

Weierstrass approximation theorem (1885)

$f \in C[a,b] \Rightarrow \exists\, (p_n)$ polynomials, $p_n \to f$ uniformly on $[a,b]$.

M.H. Stone's theorem (1948)

X: Hausdorff $\quad C(X) = C(X, \mathbb{R})$

$\mathcal{A} \subset C(X)$ algebra $\begin{cases} \text{vanishing nowhere: } (\forall p \in X)(\exists f \in \mathcal{A})\ f(p) \neq 0 \\ \text{separating points: } p, q \in X,\ p \neq q \\ \quad \Rightarrow \exists\, h \in \mathcal{A},\ h(p) \neq h(q) \end{cases}$

<u>Then</u> $\mathcal{A}$ is <u>dense</u> in $C(X)$ (note this requires X Hausdorff)

(for the u.o.c. topology).

<u>Lemma 1</u> If $\mathcal{A} \subset C(X)$ is an algebra vanishing nowhere and separating points, then for any $p \neq q$ in X, any $c_1, c_2 \in \mathbb{R}$, $\exists\, f \in \mathcal{A}$ sat. $f(p) = c_1$, $f(q) = c_2$. (*)

<u>Proof.</u> Let $g_1, g_2 \in \mathcal{A}$ with $g_1(p) \neq 0$, $g_2(q) \neq 0$. Then $g = g_1^2 + g_2^2 \in \mathcal{A}$ is nonvanishing at p and q. Let $h \in \mathcal{A}$ w/ $h(p) \neq h(q)$, and find $f \in \mathcal{A}$ of the form $f(x) = \alpha\, g(x) + \beta\, g(x) h(x)$. The 2×2 linear system for (α, β) given by $\begin{cases} f(p) = c_1 \\ f(q) = c_2 \end{cases}$ has determinant $g(p) g(q) [h(q) - h(p)] \neq 0$.

Hence there exists a unique sln (α, β).

(*) We may call this the "two point interpolation property" for $\mathcal{A}$.

<u>Lemma 2</u> $\mathcal{A} \subset C(X)$ algebra

$f \in \mathcal{A} \Rightarrow |f| \in \overline{\mathcal{A}}$ (u.o.c. closure)

Pf. of lemma 2 Let $f \in \mathcal{A}$, $\varepsilon > 0$, $K \subset X$ compact, $\|f\|_K = \sup_{x\in K} |f(x)|$

Using Weierstrass's Approximation theorem, find a polynomial $p(t)$ s.t.

$$| p(t) - |t| | < \frac{\varepsilon}{2}, \quad \forall t \text{ s.t. } |t| \le \|f\|_K$$

In part $|p(0)| < \frac{\varepsilon}{2}$, so $q(t) = p(t) - p(0)$ sat.

$$| q(t) - |t| | < \varepsilon \quad \text{if} \quad |t| \le \|f\|_K .$$

Note $g = q(f) \in \mathcal{A}$ (this needs $q(0) = 0$). For $x \in K$, ~~we~~ we have:
↳ (since we don't assume $\mathcal{A}$ includes constants)

$$| g(x) - |f(x)| | < \varepsilon .$$

This shows $|f| \in \overline{\mathcal{A}}$. □

Corollary of lemma 2 : "$\overline{\mathcal{A}}$ is a lattice" (note $\overline{\overline{\mathcal{A}}} = \overline{\mathcal{A}}$ is an algebra if $\mathcal{A}$ is.)

$$f, g \in \overline{\mathcal{A}} \longrightarrow \min\{f,g\} \in \overline{\mathcal{A}}, \ \max\{f,g\} \in \overline{\mathcal{A}} .$$

follows from $\min\{f,g\} = \frac{1}{2}[f+g-|f-g|]$, $\max\{f,g\} = \frac{1}{2}[f+g+|f-g|]$.

Lemma 3 Let $\mathcal{A} \subset C(X)$ be a subset which (i) has the 2-point interpolation property and (ii) is a lattice. Then $\mathcal{A}$ is dense in $C(X)$ (u.o.c.).

Proof. Let ~~ε>0~~ $f \in C(X)$, $K \subset X$ compact, $\varepsilon > 0$. We want $g \in \mathcal{A}$ s.t.

$$f(x) - \varepsilon < g(x) < f(x) + \varepsilon, \quad x \in K.$$

Given $p, q \in X$, let $h_{pq} \in \mathcal{A}$ satisfy $h_{pq}(p) = f(p)$, $h_{pq}(q) = f(q)$ (use the interpolation property ; $p = q$ allowed.).

Fix $p \in K$. Then $\underbrace{\ }_{\forall q \in K}$ $\exists\, U_{pq}$ nbd of q s.t. $f(x) - \varepsilon < h_{pq}(x) \quad \forall x \in U_{pq}$

So $K \subset U_{pq_1} \cup \dots \cup U_{pq_N}$, and if $h_p = \max_{1 \le i \le N} h_{pq_i} \in \mathcal{A}$, $f(x) - \varepsilon < h_p(x) \ \forall x \in K$.
$h_p(p) = f(p) \ \forall p \in K$

Now vary p :
$\exists\, V_p$ n.bd. of p s.t. $h_p(x) < f(x) + \varepsilon \quad \forall x \in V_p$

Cover $K \subset V_{p_1} \cup \dots \cup V_{p_M}$, let $g(x) = \min_{1 \le j \le M} (h_{p_j}) \in \mathcal{A}$. Then g sat :

$$f(x) - \varepsilon < g(x) < f(x) + \varepsilon \quad \forall x \in K.$$

Remark Note in lemma 3, "$\mathcal{A}$ is an algebra" is not needed.

Lemmas 1, 2, 3 prove Stone's theorem (assuming the Weierstrass Approx Thm).

Proof of Weierstrass Approximation. (may assume the interval is $[0,1]$.)

There is a "formula". For $f \in C[0,1]$, $n \geq 1$, define:

$$B_n[f] = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k} \in P_n[0,1]$$

(P_n: polynomials of degree $\leq n$) : "Bernstein polynomials".

$B_n : C[0,1] \longrightarrow P_n[0,1]$ is a bounded linear operator: $\|B_n\| = 1$

Lemma 4. $B_n[1] = 1$, $B_n[x] = x$, $B_n[x^2] = x^2 + \frac{1}{n} x(1-x)$.

Pf. (outline) Differentiate $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ in x (twice), and set $y = 1-x$ to get:

$$n = \sum_{k=1}^{n} \binom{n}{k} k x^{k-1} (1-x)^{n-k}, \quad n(n-1)x^2 = \sum_{k=2}^{n} \binom{n}{k} k(k-1) x^k (1-x)^{n-k}.$$

Exercise 1 B_n restricted to $\mathrm{span}_{\mathbb{R}}\{1, x, x^2\}$ is diagonalizable, with eigenvalues $\{1, 1, \frac{n-1}{n}\}$, eigenfunctions $1, x, x(1-x)$.

Exercise 2 B_n preserves monotonicity and convexity of functions

modulus of continuity Since $f \in C[0,1]$ is unif. cont., defining for $\delta > 0$:

$$\omega_f(\delta) = \sup\{\, |f(x) - f(y)| \,;\, x, y \in [0,1],\ |x-y| < \delta \,\}$$

$\omega_f(\delta) \downarrow 0$ as $\delta \downarrow 0$.

Let $r_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $x \in [0,1]$ $\quad r_{nk} \in P_n[0,1]$ $\quad k = 0, \dots, n$

Lemma 5 $\boxed{\| f - B_n[f] \| \leq \left(1 + \frac{1}{4n\delta^2}\right) \omega_f(\delta)}$ $\forall \delta > 0$. (sup norm).

Then given $\varepsilon > 0$ choose $\delta > 0$ so that $\omega_f(\delta) < \frac{\varepsilon}{2}$, then n so that $\frac{1}{4n\delta^2} < \frac{\varepsilon}{2}$ to get $\| f - B_n[f] \| < \varepsilon$, proving Weierstrass approx.

Cor choosing $\delta = \frac{1}{\sqrt{n}}$: $\boxed{\| f - B_n[f] \| \leq \frac{5}{4} \omega_f\left(\frac{1}{\sqrt{n}}\right)} \longrightarrow 0$ as $n \to \infty$

Proof of lemma 5. We have $f(x) - B_n[f](x) = \sum_{k=0}^{n} \left(f(x) - f(\tfrac{k}{n})\right) r_k(x)$

(Fix $n \geq 1$. and $\delta > 0$) Let $x \in [0,1]$. Set $K_1 = \{ k \mid |x - \tfrac{k}{n}| \leq \delta \}$, $K_2 = \{ k \mid |x - \tfrac{k}{n}| > \delta \}$

Then $|f(x) - B_n[f](x)| \leq \sum_{k \in K_1} |f(x) - f(\tfrac{k}{n})| r_k(x) + \sum_{k \in K_2} |f(x) - f(\tfrac{k}{n})| r_k(x)$

and the sum over K_1 is bounded above by $\omega_f(\delta)$.

To estimate the sum over K_2: divide the interval between x and $\tfrac{k}{n}$ into segments of length δ. There are $\frac{|x - \frac{k}{n}|}{\delta}$ segments, and following the change of f along points in the chain we find:

$$|f(x) - f(\tfrac{k}{n})| \leq \frac{|x - \frac{k}{n}|}{\delta} \omega_f(\delta) \leq \frac{(x - \frac{k}{n})^2}{\delta^2} \omega_f(\delta),$$

since $\left|\frac{x - \frac{k}{n}}{\delta}\right| > 1$ for $k \in K_2$. Thus the sum over K_2 is bounded above by:

$$\frac{\omega_f(\delta)}{\delta^2} \sum_{k=0}^{n} \left(x^2 - \frac{2k}{n} x + \frac{k^2}{n^2}\right) r_k(x) = \frac{\omega_f(\delta)}{\delta^2}\left(x^2 - 2x B_n[x] + B_n[x^2]\right)$$

(Lemma 4)
$$= \frac{\omega_f(\delta)}{\delta^2}\left(x^2 - 2x^2 + x^2 + \frac{1}{n} x(1-x)\right) = \frac{\omega_f(\delta)}{\delta^2} \frac{1}{n} x(1-x) \leq \frac{\omega_f(\delta)}{4n\delta^2} \quad \square$$

(Lem 5)

Remark on complex-valued functions

Let $X = \mathbb{D} \subset \mathbb{C}$ (open unit disk) $\mathcal{A}$: algebra of polynomials in $z \in \mathbb{D}$.
(nowhere vanishing, separates points)

BUT $(p_n \in \mathcal{A},\ p_n \to f$ u.o.c. in $\mathbb{D}) \Rightarrow$ f is analytic in $\mathbb{D}$!

Thus Stone's theorem is NOT TRUE for $\mathcal{A} \subset C(X;\mathbb{C})$.

But it's true if we add a requirement to $\mathcal{A} \subset C(X;\mathbb{C})$: $\boxed{f \in \mathcal{A} \Rightarrow \bar{f} \in \mathcal{A}}$.

Note $f = \mathrm{Re}\, f + i\, \mathrm{Im}\, f$, $\mathrm{Re}(f) = \frac{1}{2}(f + \bar{f})$, $\mathrm{Im}\, f = \frac{1}{2i}(f + \bar{f})$ $\mathrm{Re}\, f, \mathrm{Im}\, f \in C(X;\mathbb{R})$.

Exercise 3 The algebra $\mathcal{B} = \{ f + \bar{f};\ i(f - \bar{f});\ f \in \mathcal{A} \} \subset C(X;\mathbb{R})$ is separating and nowhere vanishing if $\mathcal{A} \subset C(X;\mathbb{C})$ is (and is an algebra).

Then given $f \in C(X;\mathbb{C})$, $\varepsilon > 0$, $K \subset X$ compact, find $g_1 = f_1 + \bar{f}_1$, $g_2 = i(f_2 - \bar{f}_2)$ $(f_1, f_2 \in \mathcal{A} \Rightarrow g_1, g_2 \in \mathcal{A})$ with $\|g_1 - \mathrm{Re}(f)\|_K < \varepsilon$, $\|g_2 - \mathrm{Im}\, f\|_K < \varepsilon$.

Let $g = g_1 + i g_2 \in \mathcal{A}$

Then $\|g - f\|_K = \sup_{x \in K} \left[(g_1 - \mathrm{Re}\, f)^2(x) + (g_2 - \mathrm{Im}\, f)^2(x) \right]^{1/2} < \sqrt{2}\, \varepsilon$.