MATH 561–FINAL EXAM (online)–December 4, 2020, 10:00–12:45. Give complete proofs of the following statements.

1. (i) If $\lim f_n(c) = L$ exists (for some $c \in \mathbb{R}$, where $f_n : \mathbb{R} \to \mathbb{R}$ is continuously differentiable for $n \geq 1$) and the sequence of first derivatives (f_n') converges to 0 uniformly on compact sets, then $f_n \to L$ uniformly on compact sets.

(ii) If $f : [0,1] \to \mathbb{R}$ is continuous and $\int_0^1 x^n f(x)dx = 0$ for each $n = 0, 1, 2, ...$, then $f(x) \equiv 0$ on $[0,1]$. *Hint:* Stone-Weierstrass, $\int_0^1 f^2(t)dt = 0$.

2. The compact-open topology in $C(X,Y)$ is regular if Y is regular. (You may assume the corresponding statement with 'Hausdorff' in place of 'regular' is true.)

3. (i) $P = X \times Y$ is connected if (and only if) both X and Y are. *Hint:* Fix $(x_0, y_0) \in P$. For $x \in X$, let $T_x = (\{x\} \times Y) \cup (X \times \{y_0\})$. Explain why T_x is connected, why this implies P (union of all $T_x, x \in X$) is connected.

(ii) Connected open sets in a locally path connected metric space are path connected.

4. (i) If X, Y are perfect metric spaces (no isolated points), then so is the product space $X \times Y$.

(ii) If X, Y are totally disconnected metric spaces, then so is the product space $X \times Y$.

(iii) Let $C \subset \mathbb{R}$ be the Cantor middle-thirds set. Explain why $C \times C$ is homeomorphic to C.

5. (i) Let $\delta > 0$. Find an open cover of $\mathbb{R}$ of order 2 (any point is in at most two sets of the cover), by open intervals of diameter δ.

(ii) Let $X \subset \mathbb{R}$ be a compact set. Then X has covering dimension at most 1. (I.e. any open cover of X admits an open refinement of order at most 2.) *Hint:* given an open cover of X, let 2δ be its Lebesgue number.

6. Let $p : X \to Y$ be a covering map. If X is path-connected and Y is simply-connected, then p is a homeomorphism.