NOTES ON BAIRE'S THEOREM

Example. A complete metric space (X, d) without isolated points is *uncountable.*

Suppose by contradiction $X = \{x_1, x_2, \ldots\}$. Let $y_1 \neq x_1$ and $0 < r_1 < 1$ be such that $x_1 \notin \bar{B}_{r_1}(y_1)$. Then choose $y_2 \in B_{r_1}(y_1)$ and $r_2 > 0$ so that $y_2 \neq x_2$ and $\bar{B}_{r_2}(y_2) \subset B_{r_1}(y_1)$, with $0 < r_2 < 1/2$. We can do this since X has no isolated points.

Proceeding in this fashion we get a descending chain of closed balls:

$$\bar{B}_{r_1}(x_1) \supset \bar{B}_{r_2}(y_2) \supset \ldots \qquad r_n < \frac{1}{n},$$

so (y_n) is Cauchy, and by completeness $y_n \to y$. But $y \neq x_n$ for all n, contradiction.

Baire's Theorem. Let (G_n) be a countable family of open dense sets in a complete metric space X. Then $\bigcap_{n\geq 1} G_n$ is dense in X (in particular non-empty.)

Informally, a property defined by an open set (within a class X of mathematical objects) is thought of as 'stable'; a property defined by a *dense* subset of X can be thought of as 'generic' (any object in X may be approximated by a sequence of objects with the property).

Definition. A Hausdorff topological space X is a *Baire space* if countable intersections of open dense subsets of X are dense. Thus Baire's theorem says complete metric spaces are Baire spaces.

Some terminology: a countable intersection of open, dense sets in a Baire space is called *residual.* Thus residual sets are dense.

By complementation, one sees that in a Baire space: a countable union of closed sets with empty interior has empty interior. This connects with some old terminology:

A subset P of a space X is *nowhere dense* (or 'of first category') if for any $U \subset X$ open, there exists $V \subset U$ containing no elements of P: $P \cap V = \emptyset$.

Exercise 1. Let X be a Hausdorff space. Show that P is nowhere dense if and only if its closure $\overline{P}$ has empty interior.

Proof. (i) Assume P is nowhere dense. Let $x \in \overline{P}$, U_x open nbd of x, $V \subset U_x$ open in X so that $V \cap P = \emptyset$. Since V is open, also $V \cap \overline{P} = \emptyset$. Thus x is not an interior point of $\overline{P}$. (ii) Conversely, assume $int(\overline{P}) = \emptyset$. Let $U \subset X$ open, $x \in P \cap U$ not an interior point of $\overline{P}$. Thus $\exists z \in (\overline{P})^c \cap U$, and since $(\overline{P})^c$ is open, we find a neighborhood of z, $V \subset U \cap (\overline{P})^c$. This shows P is nowhere dense.

Here are two applications to Analysis:

Example. There exists a function $f \in C[0,1]$ which is not monotone on any interval. In fact the set of such functions is *residual* in $C[0,1]$ (endowed with the sup norm.)

Idea. To see this, let $(I_n)_{n\geq 1}$ be an enumeration of the set of subintervals of $[0,1]$ with endpoints in $\mathbb{Q}$. Let E_n be the set of $f \in C[0,1]$ which are *not* monotone on I_n. The idea is to show that E_n is open and dense, and apply Baire's theorem.

E_n is *open*: if $f \in E_n$, we may find $x < y < z$ in E_n so that $f(x) < f(y)$ and $f(z) < f(y)$ (the other case, f dropping between x and z, is similar.) Then if $||f-g|| < \frac{1}{2}\min\{f(y)-f(x), f(y)-f(z)\}$, it is easy to see that g also fails to be monotone, in the same way as f.

E_n is *dense*: let $f \in C[0,1]$, and say f is monotone increasing on I_n. Pick $x \in I_n$. Given $\epsilon > 0$, we may find $x_- < x < x_+$ very close to x, so that $f(x+)$ and $f(x-)$ are ϵ-close to $f(x)$. Then we can change f slightly in the interval (x_-, x_+) (and nowhere else), to find g continuous and ϵ-close to f in sup norm, so that g is not monotone on this interval (say $g(x_-) > g(x)$ and $g(x_+) > g(x)$), hence not on I_n.

Example. *Uniform Boundedness Theorem.* Let E, F be Banach spaces, and consider a family of linear maps $T_\alpha \in \mathcal{L}(E,F)$, $\alpha \in \Lambda$. If the family is *equibounded* at each $x \in E$ ($||T_\alpha(x)|| < M(x)$ for all $\alpha \in \Lambda$), then it is *uniformly equicontinuous* on E:

$$\sup_{\alpha\in\Lambda} ||T_\alpha|| < \infty.$$

Proof. For each $n \geq 1$, consider the closed subset of E:

$$E_n = \{x \in E; ||T_\alpha(x)|| \leq n \forall \alpha \in \Lambda\}.$$

By assumption $\bigcup_{n\geq 1} E_n = E$. Thus by Baire's theorem, some E_{n_0} has nonempty interior: $x_0 \in E_{n_0}$ and $||x - x_0|| < r \Rightarrow x \in E_{n_0}$. Let $\sup_{\alpha\in\Lambda} ||T_\alpha(x_0)||\} = c$. Then if $x \in E$, $||x|| < 1$, we have $x_0 + rx \in E_{n_0}$, thus for arbitrary $\alpha \in \Lambda$:

$$||T_\alpha(x)|| = \frac{1}{r}||T_\alpha(x_0 + rx) - T_\alpha(x_0)|| \leq \frac{n_0 + c}{r},$$

concluding the proof.

Theorem. Locally compact Hausdorff topological spaces X are Baire spaces.

Proof. Let $G_1, G_2, \ldots$ be open dense sets. Let $U \subset X$ be open. Then $U \cap G_1 \neq \emptyset$, and $\exists B_1$ open, with compact closure, so that $\bar{B_1} \subset U \cap G_1$. In the same way, we successively find open sets B_n with compact closure, so that $\bar{B_n} \subset B_{n-1} \cap G_n$.

The $\bar{B}_n$ are closed in the compact $\bar{B}_1$ and nested, so $\bigcap_{n\geq 1} \bar{B}_n \neq \emptyset$, and this intersection is contained in $U \cap \bigcap_{n\geq 1} G_n$ (since $\bar{B}_n \subset G_n \forall n$, and $\bar{B}_1 \subset U \cap G_1$. Thus $(\bigcap_{n\geq 1} G_n) \cap U \neq \emptyset$, as we wished to show.

Proposition. If X is a Baire space, any open subset $U \subset X$ is also a Baire space.

Definition. A subset of a topological space is a G_δ set if it is a countable intersection of open sets.

Example. In a metric space (X, d), any closed set A is a G_δ, since

$$A = \bigcap_{n\geq 1} G_n, \quad G_n = \{x \in X; d(x, A) < \frac{1}{n}\}.$$

Example. The set of rational numbers $\mathbb{Q} \subset \mathbb{R}$ *is not* a G_δ set. If it were, we'd have:

$$\mathbb{Q} = \bigcap_{n\geq 1} G_n,$$

with each G_n open and also dense. (Any open subset of $\mathbb{R}$ intersects $\mathbb{Q}$, hence would intersect each G_n.) But then we can add to the countable family (G_n) of open dense sets the countable family $\{r_n\}^c_{n\geq 1}$ (complement of the one-point sets $\{r_n\}$, where the r_n are an enumeration of $\mathbb{Q}$.) Since each of these sets is open and dense in $\mathbb{R}$, taken together these families would necessarily have nonempty intersection (by Baire's theorem). But clearly the intersection is empty.

What this argument shows is that no countable dense set without isolated points can be a G_δ (in a complete metric space, or a locally compact space.)

Example. Let X be a topological space, Y a metric space, $f : X \to Y$ any map. Then the set of continuity C_f of f is a G_δ (which may be empty!)

Indeed, f is continuous at $p \in X$ iff $\forall n \geq 1$ $\exists U$ nbd of p so that $d(f(x), f(y)) < 1/n$ $\forall x, y \in U$. Set:

$$A_n = \{p; \exists U \text{ nbd of } p; d(f(x), f(y)) < \frac{1}{n} \forall x, y \in U\}.$$

Considering the family of open sets of X:

$$\Lambda_n = \{U \text{ open };d(f(x),f(y)) < \frac{1}{n} \forall x,y \in U\}$$

we have that A_n is the union of this family, an open subset of X:

$$A_n = \bigcup\{U; U \in \Lambda_n\}$$

and clearly:

$$C_f = \bigcap_{n \geq 1} A_n,$$

and hence C_f is a G_δ.

Example. In particular, $\mathbb{Q}$ cannot be the set of continuity of a function from $\mathbb{R}$ to $\mathbb{R}$. But the irrationals $\mathbb{I}$ can be. For example, *Thomae's function*:

$$f(x) = \frac{1}{q}, x = \frac{p}{q} \in \mathbb{Q}, \text{ with } p \in \mathbb{Z}, q \in \mathbb{N} \text{ coprime}; \quad f(x) = 0, x \in \mathbb{I}$$

is continuous exactly at points of $\mathbb{I}$.

Exercise 2. Let $f_n : X \to Y$ be continuous (X topological, (Y,d) metric.) Suppose $f_n \to f$ pointwise on X. Then each level set $\{x \in X; f(x) = L\}$ of f is a G_δ subset of X.

Proof: $x \in X$ is in the level set if, and only if, there exists a subsequence of $(f_n(x))$ converging to L. Thus the level set Z equals:

$$Z = \bigcap_{j \geq 1} \bigcap_{i \geq 1} \bigcup_{n \geq i} f_n^{-1}(B(L, \frac{1}{j})),$$

where $B(L, \frac{1}{j}) = \{y \in Y; d(y,L) < \frac{1}{j}\}$. Since each f_n is continuous and $\mathbb{N} \times \mathbb{N}$ is countable, this set is a G_δ.

Note $\mathbb{Q} \subset \mathbb{R}$ is dense, but not residual; while its complement $\mathbb{I}$ (irrationals) is a residual subset of $\mathbb{R}$ (why?) Above we outlined a proof that the set of functions in $C[0,1]$ which are not monotone on any interval of $[0,1]$ is residual. Here is another example:

Example: The set of functions in $C[0,1]$ which are not differentiable at any point is residual (in the uniform topology.)

Outline of proof: (for details, see [Munkres 1, no. 49]). For $f \in C[0,1]$, $0 < h < 1/2$ and $x \in I = [0,1]$, define:

$$\Delta_f(x,h) = \max\{|\frac{f(x+h) - f(x)}{h}|, |\frac{f(x) - f(x-h)}{h}|\}.$$

(If $0 < x < 1/2$, at least one of these difference quotients is defined; if only one is defined, take $\Delta_f(x,h)$ to be that one.). Then set:

$$\Delta_f(h) = \inf\{\Delta_f(x,h); x \in I\}.$$

We consider the set of $f \in C[0,1]$ that have large slope on small intervals:

$$U_n = \bigcup_{0<h<1/n} \{f \in C[0,1]; \Delta_f(h) > n\}.$$

And then prove the following claims:

(i) U_n is open in $C[0,1]$;

(ii) U_n is *dense* in $C[0,1]$ (this is the hard part.)

(iii) If $f \in \bigcap_{n\geq 1} U_n$, f is nowhere differentiable.

Remark: This should be contrasted with the fact that, by the Stone-Weierstrass theorem, polynomials are dense in $C[0,1]$; in particular smooth functions are dense.

An early application of Baire's theorem is the following:

Theorem. Let X be a Baire space, Y a metric space. If $f : X \to Y$ is the pointwise limit of continuous maps $f_n : X \to Y$, then the continuity set $C_f \subset X$ is residual in X.

Proof (outline). For $\epsilon > 0, N \geq 1$, define

$$A_N(\epsilon) = \{x \in X; d(f_m(x), f_n(x)) \leq \epsilon, \forall x \in X.\}$$

The $A_N(\epsilon)$ are closed in X (since the f_n are continuous.) Since $(f_n(x))$ is a Cauchy sequence in Y for any x, we have $\bigcup_{N\geq 1} A_N(\epsilon) = X$. Now let

$$U(\epsilon) = \bigcup_{N\geq 1} int(A_N(\epsilon))$$

(interior in X). $U(\epsilon)$ is evidently open in X. Now consider the lemma, whose proof uses the fact that open subsets of Baire spaces are Baire spaces:

Lemma. Let X be a Baire space, $F_n \subset X$ closed subsets so that $X = \bigcup_{n\geq 1} F_n$. Then $A = \bigcup int(F_n)$ is (open and) dense in X.

Thus the set $U(\epsilon)$ is dense in X, and therefore $S = \bigcap_{k\geq 1} U(\frac{1}{k})$ is residual in X. The following claim is not hard to prove, and concludes the proof of the theorem:

Claim: $S \subset C_f$.

Historical remark. This theorem led Baire to define 'classes' of functions $f : R \to R$, as follows: continuous f are 'class 0'. Pointwise limits of class 0 functions which are not themselves class 0 are 'class 1' (examples are easy to find.) Pointwise limits of class 1 functions which are not themselves class 1 are 'class 2'; and so on. For example, consider the function:

$$D(x) = \lim_{k\to\infty} \{ \lim_{j\to\infty} (\cos(k!\pi x))^j \}.$$

Exercise 3. Show that $D(x) = 1$ for $x \in \mathbb{Q}$, $D(x) = 0$ for $x \in \mathbb{I}$. Use this to show that D is in Baire's 'class 2'.

H.Lebesgue (in 1904) proved that each Baire class is nonempty, and that functions exist which are in no Baire class.

The following example is in the same circle of ideas:

Example: Let $f : R \to R$ be a differentiable function. Then the continuity set of f' is residual in R.

Indeed, f' is the pointwise limit of continuous functions:

$$f'(x) = \lim_{k\to\infty} k[f(x + \frac{1}{k}) - f(x)].$$