MATH 561 - 9/1/2022

Urysohn metrization thm

X normal 2^{nd} ctble Hausdorff $\Rightarrow \exists\ F: X \longrightarrow C$ (Hilbert cube)

$$C = \left\{ \bar{x} = (x_i)_{i\geq 1} \in \ell^2 \;\middle|\; 0 \leq x_i \leq 1/i\ ,\ i \geq 1 \right\}$$

(w/ metric induced from $\|\bar{x} - \bar{y}\|^2 = \sum_{i\geq 1} (x_i - y_i)^2$) in ℓ^2

(C homeo to $[0,1]^{\mathbb{N}}$ w/ product top.).

Lemma

$\mathcal{B} = (B_n)$ ctble basis for X

$P \subset \mathbb{N}$ ctble set of pairs (B_n, B_m) s.t. $\overline{B_n} \subset B_m$

$f_i : X \longrightarrow [0,1]$ cont. $f_i|_{\overline{B}_n} \equiv 1$, $f_i \equiv 0$ on B_m^c

Lemma $\forall x \in X\ \exists\ i \in P$ s.t. $\begin{cases} f_i(x) = 1 \\ f_i|_{U^c} = 0 \end{cases}$
$\forall U_x$

(f_i : Urysohn fn. for (B_n, B_m)).

Let $F(x) = \left(f_1(x), \frac{1}{2} f_2(x), \frac{1}{3} f_3(x), \dots \right) \in C$

We have $F: X \longrightarrow C$ cont. (since each component $\pi_i \circ F$ is)

If $x \neq y \in X$ $\exists\ U_x$ s.t. $y \in U_x$

$\exists\ i \geq 1$ s.t. $f_i(x) = 1$, $f_i(y) = 0$. So $F(x) \neq F(y)$.

Claim F is homeo onto its image $F(X) \subset C$ (w/ induced topology)

E.T.S. F is an open map onto $F(X)$

(i.e. $U \subset X$ open $\Rightarrow F(U)$ open in $F(X)$, i.e. $F(U) = A \cap F(X)$, A open in C)

we may assume $U = B_k \in \mathcal{B}$

$F: X \longrightarrow C \qquad U = B_k \subset X$

Let $\quad J = \{ i \in \mathbb{N} ; \exists P_i = (B_m, B_k) \text{ w/ } \overline{B}_m \subset B_k \}$

$A_k = \bigcup_{i \in J} \{ \overline{x} \in C ; x_i > 0 \}$ open in C

<u>Claim</u> $\quad F(B_k) = F(X) \cap A$

1) Let $x \in B_k$ $\quad \exists\, m$ s.t. $x \in B_m$, $\overline{B}_m \subset B_k$ so $P_i = (B_m, B_k)$

$f_i(x) = 1 \quad (f_i \equiv 1 \text{ on } \overline{B}_k)$ sat $\overline{B}_m \subset B_k$, $i \in P$

so $F(x) \in A_k \cap F(X)$.

2) conv. if $F(x) \in F(X) \cap A_k$ (for some $x \in X$)

so $F(x)_i = f_i(x) > 0$ for some $i \in J$

i.e. $\exists \underbrace{(B_m, B_k)}_{P_i}$ w/ $\overline{B}_m \subset B_k$ and $f_i(x) > 0$

$(f_i(x) = 0 \text{ on } B_k^c)$ so $x \in B_k$

<u>i.e.</u> $\quad y \in F(X) \cap A_k \Rightarrow y = F(x)$ for some $x \in B_k$.

or $\quad F(B_k) = F(X) \cap A_k$. $\qquad$ so <u>F is an embedding</u>

<u>Remark</u>

If $\exists\, F: X \longrightarrow C$ (Hilbert cube), then X is metrizable, and 2^{nd} ctble (in part. <u>separable</u>)

<u>Conversely</u> if X is ~~not~~ a separable metric sp, then $\exists$ an embedding $F: X \longrightarrow C$

Let $\quad F_i(x) = \dfrac{1}{i} \dfrac{d(x, B_i^c)}{1 + d(x, B_i^c)}$ where (B_i) is a ctble basis of open sets.

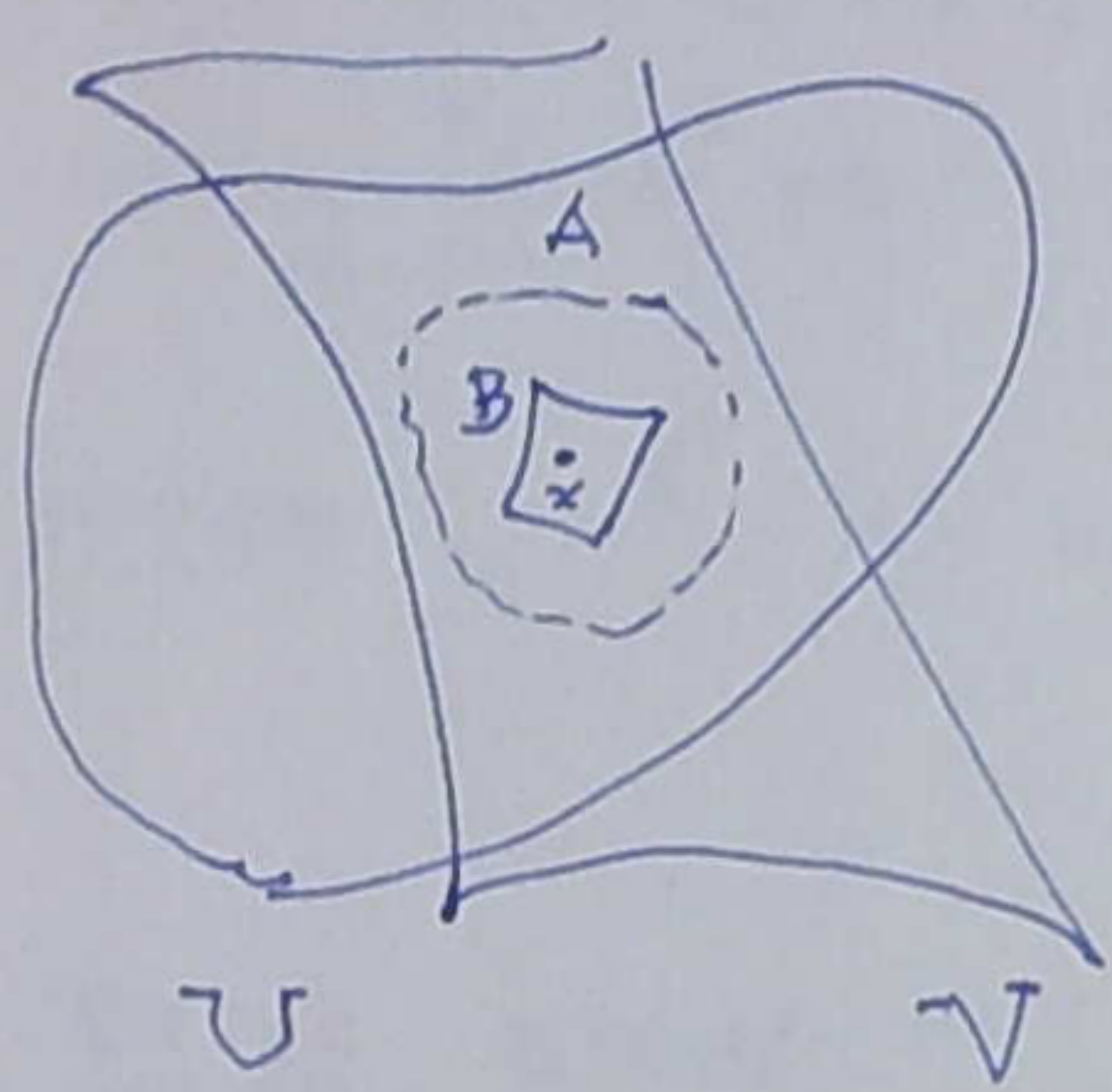
<u>(exam)</u>

<u>Prop.</u> X <u>loc. cpt.</u> Hausdorff $\Longrightarrow$ X regular.

(<u>recall</u> X regular if each pt. admits a "fund. system of ~~nbds~~ closed nbds".
$\forall x \in X$, U open nbd of x $\exists$ V open nbd of x s.t. $\overline{V} \subset U$)

<u>enough to show</u> compact nbds constitute a fund. system at each pt.)

<u>Proof</u> Let $x \in X$, U open nbd of x, <u>want</u> cpt. nbd of x <u>contained in U</u>



Let V : <u>cpt. nbd</u> of x. (from def. of loc. cpt.)
($x \in \operatorname{int} V$)

$\exists$ $A \subset U \cap V$ open in X, $x \in A$

V is compact Hausdorff. hence <u>normal</u> (in part. regular)

$\exists$ <u>B open in V</u> s.t. $x \in B$, $\overline{B} \subset A$

($\overline{B}$ closed, $\overline{B} \subset V$ compact, so $\overline{B}$ <u>compact</u>

$B = W \cap V$ (W open in X)
$\quad = W \cap A$ $\quad (x \in B \Rightarrow x \in W$ and $x \in A)$

so B is <u>open in X</u>.

<u>Prop</u> X regular + 2^{nd} ctble $\Longrightarrow$ X normal

<u>Pf.</u> $F, G \subset X$ closed disjoint.

Given $x \in F$ $\exists\, U_x$ s.t. $\overline{U_x} \cap G = \emptyset$ (since X regular)

$\{U_x\}_{x \in F}$ admits a <u>ctble</u> subcover $F \subset \bigcup_{n \geq 1} U_n$ (Lindelöf property of 2^{nd} ctble spaces)

$\overline{U}_n \cap G = \emptyset$

Likewise $G \subset \bigcup_{m \geq 1} V_m$, $\overline{V}_m \cap F = \emptyset$

Let $A_n = U_n \setminus (\overline{V}_1 \cup \dots \cup \overline{V}_n)$ (open)

$B_n = V_n \setminus (\overline{U}_1 \cup \dots \cup \overline{U}_n)$ (open

<u>note</u> $F \subset \bigcup_{n} A_n$ $\qquad G \subset \bigcup_{n \geq 1} B_n$

$x \in F \Rightarrow x \in U_{n_0}$ for some n_0, $x \notin \overline{V}_j$ $j = 1, \dots, n_0$ so $x \in A_{n_0}$.

<u>claim</u> $A_n \cap B_m = \emptyset \quad \forall n, m.$

<u>say</u> $A_n \cap B_{n+p} = \emptyset \quad (p \geq 0)$

Pf Let $x \in A_n$. Then $x \in U_n$

if $x \in B_{n+p}$: $x \notin \bar{U}_1, x \notin \bar{U}_2, \ldots x \notin \overline{U_{n+p}}$ (in part $x \notin U_n$)

so $A = \bigcup A_n$, $B = \bigcup B_n$ are disjoint open sets containing F, G resp..

<u>Conclusion</u> X top. manifold $\Longrightarrow$ X normal, 2nd ctble
$\Longrightarrow$ X metrizable.

<u>Tietze</u> <u>extension</u> thm.

X normal, $f: C \longrightarrow [-1, 1]$ cts.
$C \subseteq X$ closed

Then $\exists$ $\bar{f} : X \longrightarrow [-1, 1]$ cont extension of f.