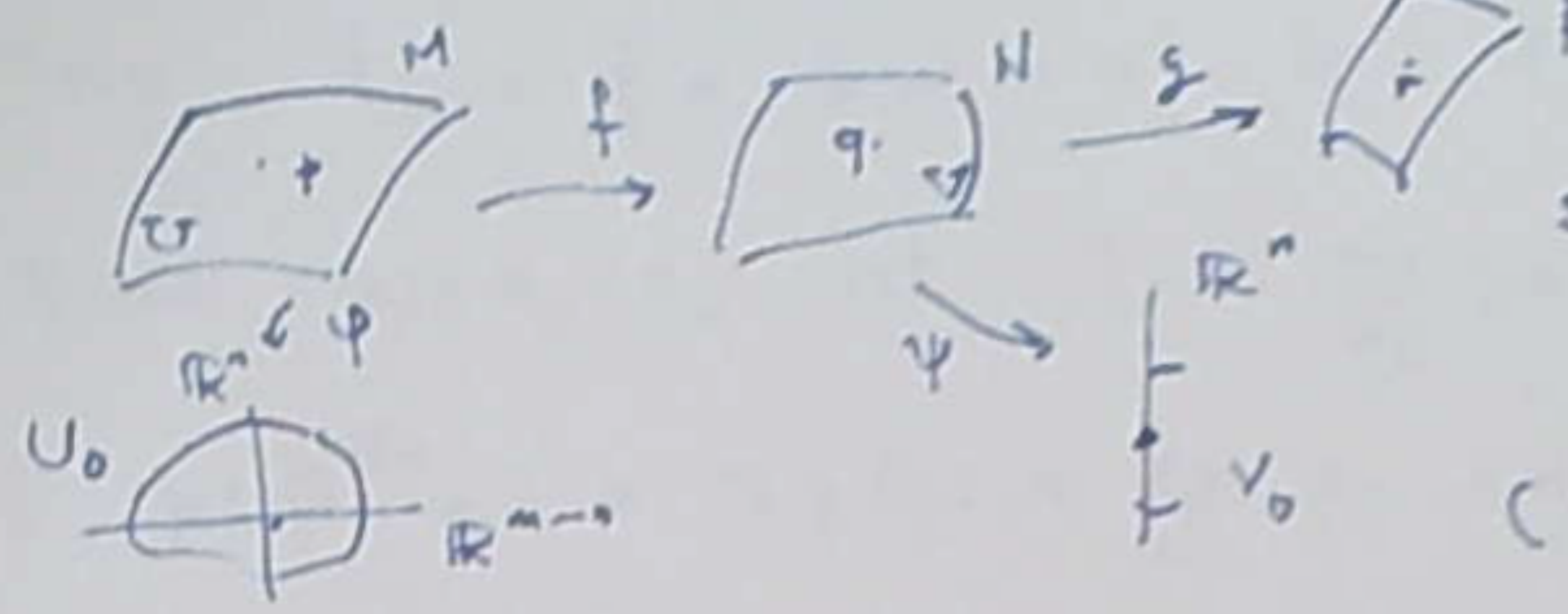
Problem set 3 – solutions

$\boxed{1}$ $f: M^m \to N^n$ C^k submersion (onto) ; $g: N \to P$ cont.

Assume $g\circ f: M \to P$ is C^k. Then g is C^k.

Let $q \in N$, $p \in M$ s.t. $f(p) = q$ (using f onto)

charts $V \subset N$ nbd of q, $\psi: V \to \mathbb{R}^n$, $\varphi: U \to \mathbb{R}^m = \mathbb{R}^{m-n} \times \mathbb{R}^n$, $f(U) = V$



$\psi(V) = V_0 \subset \mathbb{R}^n$ (by continuity)

s.t. $\psi\circ f\circ \varphi^{-1}(x,y) = y$

for $(x,y) \in U_0 \subset \mathbb{R}^{m-n} \times \mathbb{R}^n$

(local form of subm.)

w/ $g(q) = r \in P$

on U:

$(g\circ f)(x) = (g\circ \psi^{-1}) \circ (\psi\circ f\circ\varphi^{-1})(\varphi(x))$ $\quad x \in U \quad \varphi(x) = (\varphi^1(x), \varphi^2(x)) \in \mathbb{R}^{m-n} \times \mathbb{R}^n$

$= \cancel{\quad} (g\circ\psi^{-1})(\varphi^2(x))$

or $(g\circ f)(\varphi^{-1}(x,y)) = \overset{(g\circ\psi^{-1})}{\cancel{g}}(y)$ for $(x,y) \in U_0$. $y \in V_0$ (take $U_0 = W_0 \times V_0$, $W_0 \subset \mathbb{R}^{m-n}$)

Since $\varphi^{-1}: U_0 \to U$ is a diff., $g\circ\psi^{-1}$ is C^k in V_0 if ~~g is~~ $(g\circ f)$ is C^k in U. Hence g is C^k in $V \subset N$.

(Note: given $\overset{f(q)}{\overset{\shortparallel}{r}} \in P$, ~~we fix~~ and a nbd of r (say $W \subset P$), we first find a nbd V_1 of q s.t. $g(V_1) \subset W$ (using continuity), then $V \subset V_1$ as above s.t. $g: V \to W$ is C^k).

$\boxed{2}$ $f: M \to N$. $S \subset N$ submanifold, $f \pitchfork S$.

Then $V = f^{-1}(S) \subset M$ is a submanifold, $\operatorname{codim}_M V = \operatorname{codim}_N S = \ell$

Let $p \in V$, $q = f(p) \in S$. Then $\exists \begin{cases} \varphi: W \to \mathbb{R}^\ell, \ \varphi(q) = 0 \in \mathbb{R}^\ell, \\ U \subset M \text{ nbd of } p, \ f(U) \subset W \end{cases}$

s.t. $\varphi^{-1}(0) = S \cap W$, $(\varphi\circ f): U \to \mathbb{R}^\ell$ is submersion and $(\varphi\circ f)^{-1}(0) = U \cap V$, with 0 a reg. value of $\varphi\circ f$.

so $T_p V = \operatorname{Ker}[d(\varphi\circ f)(p)]$, ~~with~~ while $T_q S = \ker[d\varphi(q)]$

Note $d(\varphi\circ f)(p) = d\varphi(q)\, df(p)$ so if ~~+~~ $df(p)\, v \in T_q S$, then $v \in T_p V$ (w/ $v \in T_pM$)

(i.e. $df(p)^{-1}[T_q S] \subset T_p V$)

~~Now~~ Conversely if $v \in T_p V$ $\quad$ ($\dim T_q S = \dim N - \ell$, since $d\varphi(q)$ is onto)

we have $d\varphi(q)\, df(p)[v] = 0$, so $df(p)[v] \in \ker d\varphi(q) = T_q S$. So the subspaces coincide.

4 We have $\sum_{i=1}^{k} x^i \frac{\partial p}{\partial x^i}(x) = m\, p(x)$ for $x \in \mathbb{R}^k$.

Thus all $a \neq 0$ in $\mathbb{R}$ are regular values of p: if $p(x) \neq 0$,
at least one $\frac{\partial p}{\partial x^i}(x) \neq 0$, so $dp(x) \in L(\mathbb{R}^k, \mathbb{R})$ is onto.
Thus $L_a = \{x;\ p(x) = a\}$ is a $(k-1)$-dim'l submanifold of $\mathbb{R}^k$.
Given $a_1, a_2 > 0$, let $\lambda = \frac{a_2}{a_1} > 0$. Then if $x \in L_{a_1}$, $\lambda^{1/m} x \in L_{a_2}$
$\left(p(\lambda^{1/m} x) = \lambda p(x) = \frac{a_2}{a_1} \cdot a_1 = a_2 \right)$ $x \mapsto \lambda^{1/m} x$ is a diff. of $\mathbb{R}^k$,
which restricts to a diff $L_{a_1} \to L_{a_2}$. Similar for $a_1 < 0$, $a_2 < 0$

5 (a) Consider $\det_n : M_n \to \mathbb{R}$

remove i^{th} row, j^{th} column from A

We have $\det_n(A) = \sum_{i,j} (-1)^{i+j} a_{ij} (\det_{n-1} A_{ij})$
so $\frac{\partial \det_n}{\partial a_{ij}}(A) = (-1)^{i+j} \det_{n-1}(A_{ij})$ and A is a crit. point
iff all $\det_{n-1}(A_{ij})$ are zero (in which case $\det A = 0$)
In particular 1 is a regular value of $\det_n$, so $SL(n) = \det_n^{-1}(1)$
is an (n^2-1) dim'l submanifold of $M_n \approx \mathbb{R}^{n^2}$.

(b) $T_I SL_n = \ker\left[d(\det)_n(I)\right]$ Note $\det(I_{ij}) = \begin{cases} 1 & i=j \\ 0, & i \neq j \end{cases}$

For $H \in M_n$: $d(\det_n)(I)[H] = \sum_{i,j} \frac{\partial \det_n}{\partial a_{ij}}(I)\, h_{ij} = \sum_i h_{ii} = tr(H)$
$H \in \ker[d(\det_n)(I)] \Leftrightarrow tr\, H = 0$. $T_I SL_n = \{H \in M_n;\ tr\, H = 0\}$

~~6~~7 (a) Δ and W_A are both n-dimensional subspaces of $V \times V$.
Thus $\Delta + W_A = V \times V \Leftrightarrow \Delta \cap W_A = \{0\} \Leftrightarrow (\nexists\, w \neq 0)(Aw = w)$
$\Leftrightarrow$ 1 is not an eigenvalue of A

~~5~~ **6** $\mathcal{M} = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \subset M_2 \setminus \{0\} \,\middle|\, \det A = 0 \right\}$ (rank 1 2x2 matrices)
$\det(A) = ad - bc \Rightarrow d\det(A)\begin{bmatrix} x & y \\ z & w \end{bmatrix} = aw + dx - bz - cy$
so vanishes for all $(x,y,z,w) \in \mathbb{R}^4$ iff $a = b = c = d = 0$, i.e. $A = 0$
Thus $\det$ is a submersion at all points of $M_2 - \{0\}$.
In particular $\mathcal{M} = \det^{-1}(0) \cap (M_2 \setminus \{0\})$ is a 3-dim'l submanifold of $M_2 - \{0\}$

$\boxed{7}$ (b) Consider the diagonal submanifold $\Delta \subset X \times X$ (codimension $n = \dim X$)

The set of fixed points of f is the preimage of Δ under the graph embedding $G_f : X \longrightarrow X \times X$, $G_f(x) = (x, f(x))$

$$\mathrm{Fix}(f) = G_f^{-1}(\Delta).$$

Thus if $G_f \pitchfork \Delta$, $\mathrm{Fix}(f)$ will be a submanifold of X of codimension n, i.e. dimension 0, i.e. $\mathrm{Fix}(f)$ consists of isolated points (finitely many, if X is compact.).

Now $dG_f(x)[v] = (v, df(x)[v]) \in T_x X \times T_x X$

if $G_f(x) \in \Delta$ (i.e. if $x \in \mathrm{Fix}(f)$).

And $T_{(x,x)}\Delta = \Delta_{T_x X} \subset T_x X \times T_x X$, the diagonal subspace

As seen in (7a),

since $dG_f(x)[T_x X] = W_{df(x)} \subset T_x X \times T_x X$, the ~~diagonal~~ graph subspace of $df(x) \in L(T_x X)$

$$dG_f(x)[T_x X] + \Delta_{T_x X} = T_x X \times T_x X \text{ iff}$$

$$W_{df(x)} + \Delta_{T_x X} = T_x X \times T_x X$$

iff 1 is not an eigenvalue of $T_x X$. Thus $G_f \pitchfork \Delta$ iff f is a Lefschetz map, and then $\mathrm{Fix}(f)$ consists of isolated points.

$\boxed{13}$ (a) A submersion $f : X \to Y$ is an open map. Thus if X is compact $f(X)$ is at once compact (hence closed) and open in Y, so $f(X) = Y$ if Y is connected.

(b) Follows from (a), since $\mathbb{R}^n$ is connected but not compact, so $f(X) = \mathbb{R}^n$ is not possible if X is compact.