MATH 561, FALL 2022-MIDTERM EXAM.

1. (i) Let. $f : M \to N$ be a continuous map between non-compact manifolds. Show that if f is proper (def: preimage of compact is compact), then f is a closed map.

(ii)Prove: If f is a smooth injective immersion from M to N (noncompact manifolds) and f is proper, then f is an embedding.

2. (i) Let $f : M \to N$ be a smooth map, $q \in N$ a regular value of f, $S = f^{-1}(q)$ (assumed non-empty), $x \in S$. Prove that for the tangent space we have:

$$T_x S = Ker(df(x)).$$

Hint: prove it for smooth maps of euclidean space first.

(ii) The space of rank one 2×2 matrices (that is, nonzero matrices with determinant zero) is a 3-dimensional submanifold of $M_2 = \mathbb{R}^4$. Find its tangent space at

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{1}$$

3. (i) Consider the open cover $\{U, V\}$ of the real line, where $U = (-\infty, 1), V = (0, +\infty)$. Show there does not exist a partition of unity with compact supports, strictly subordinate to this cover.

(ii) Let M be a connected smooth manifold. Prove that given $p, q \in M$, there exists a piecewise C^1 path from p to q.

4. let X and Z be transversal submanifolds of the manifold Y. (Recall this means the inclusion map $i : X \to Y$ is transversal to Z.) Prove that $X \cap Z$ is a submanifold of Y (of what dimension?), and that if $x \in X \cap Z$:

$$T_x(X \cap Z) = T_x X \cap T_x Z.$$

5. (i) Define 'null set' on a manifold, and explain why the definition makes sense.

(ii) Let $f : \mathbb{R} \to \mathbb{R}^2$ be a Lipschitz map (for the euclidean norms.) Prove that $f(\mathbb{R})$ is a null set in $\mathbb{R}^2$. (*Hint:* Do the case of a Lipschitz map $f : [a, b] \to \mathbb{R}^2$ first.)