MATH 561, FALL 2022–TOPOLOGY I–FINAL EXAM
Friday, December 9, 2022, 10:30–12:45. Closed book, closed notes.
This test consists of **FIVE** problems.

NAME:

1. Let M, N be smooth non-compact manifolds. Let $(x_i)_{i\geq 1}$ be a sequence in M. We say $x_i \to \infty$ if given any $K \subset M$ compact we may find $n_0 \geq 1$ so that $i \geq n_0 \Rightarrow x_i \in M \setminus K$.

(i) Prove that if $f : M \to N$ is a smooth proper map, then $x_i \to \infty$ on M implies $f(x_i) \to \infty$ on N.

(ii) Prove that if $f : M \to R^n$ is a smooth injective immersion and $\phi : M \to R_+$ is a smooth proper function, then $g(x) = (f(x), \phi(x))$ defines a smooth injective immersion from M to R^{n+1}, which in addition is a proper map.

2. (a) Let M be a smooth manifold, $g : M \to R^n$ be a smooth map $(dim(M) > n)$, $q \in R^n$ a regular value of g, $S = g^{-1}(q)$ (assumed non-empty), $x \in S$. Prove that for the tangent space we have:

$$T_xS = Ker(dg(x)) \subset T_xM.$$

(b) Let $f : M \to N$ be a C^k map, transversal to a submanifold $V \subset N$ of the smooth manifold N. Let $S \subset M$ be the submanifold $S = f^{-1}(V)$, assumed non-empty. Then if $x \in S$, prove the tangent space T_xS is:

$$T_xS = (df(x))^{-1}[T_{f(x)}V],$$

the preimage under the differential of f.

3. (i) Let $f : \mathbb{R} \to \mathbb{R}^2$ be a Lipschitz map (for the euclidean norms.) Prove that $f(\mathbb{R})$ is a null set in $\mathbb{R}^2$. (*Hint:* Do the case of a Lipschitz map $f : [a, b] \to \mathbb{R}^2$ first.)

(ii) Let $Q = \{r_1, r_2, \ldots\}$ be an enumeration of the rational numbers, and let A_{ij} be the open interval with center r_i and length $1/2^{i+j}$, for each $i, j \geq 1$. Show that the set:

$$A = \bigcap_{j\geq 1} \bigcup_{i\geq 1} A_{ij}$$

is a residual subset of the real line, and also a null set.

4.(i) Let $\delta > 0$. Describe an open cover of $\mathbb{R}$ of order 2 (any point is in at most two sets of the cover), by open intervals of diameter δ.

(ii) Let $X \subset \mathbb{R}$ be a compact set. Prove (from the definitions and part (i)) that X has covering dimension at most 1. (I.e. any open cover of X admits an open refinement of order at most 2.) *Hint:* given an open cover of X, let 2δ be its Lebesgue number.

5. Let X be a non-compact manifold, $X = \bigcup_{i \geq 1} X_i$ a compact exhaustion: ($X_i \subset X$ compact, $X_i \subset int(X_{i+1})$). Suppose (Y, d) is a bounded, complete metric space ($d(y, y') < 1$, for all $y, y' \in Y$).

(i) Consider the metric on the space of all maps $f : X \to Y$:

$$\rho(f, g) = \sum_{i=1}^{\infty} \frac{1}{2^i} \sup_{x \in X_i} d(f(x), g(x)).$$

Let $f_n \in C(X, Y)$, $f : X \to Y$. Show that $f_n \to f$ uniformly on compact sets if, and only if, $\rho(f_n, f) \to 0$.

(ii) Prove that if $f_n \to f$ pointwise in X and $\mathcal{F} = \{f_1, f_2, \ldots\} \subset C(X, Y)$ is equicontinuous (pointwise on X), then $f \in C(X, Y)$ and $f_n \to f$ uniformly on compact sets.