MATH 562, SPRING 2023–HOMEWORK SET 5

1. Prove that the boundary orientation of $S^k = \partial B^{k+1}$ is the same as its preimage orientation under the map:

$$g : R^{k+1} \to R, \quad g(x) = |x|^2.$$

2.(i) Let $f : R^2 \to R$ be a smooth function, $S \subset R^3$ its graph, $S = \{(x, y, f(x, y))\}$. Determine the preimage orientation of S under the map:

$$F : R^3 \to R, \quad F(x, y, z) = z - f(x, y).$$

(ii) A basis of tangent vectors and the upward normal at a point of the graph are given by:

$$v_1 = (1, 0, f_x), \quad v_2 = (0, 1, f_y), \quad (-f_x, -f_y, 1).$$

Is $\{v_1, v_2\}$ positively oriented? Does the preimage orientation coincide with that induced on S by the upwards or the downwards normal?

3. Prove that every compact hypersurface in euclidean space is orientable. (You may use the Jordan-Brower theorem.)

4. (i) Compute the degree of the antipodal map of S^k.

(ii) Prove the antipodal map is homotopic to the identity if and only if k is odd.

(iii) Prove there exists a nonvanishing vector field on S^k if and only if k is odd.

5. (i) Prove that for given $f : X \to Y$, $g : Y \to Z$, $deg(g \circ f) = deg(f)deg(g)$.

(ii) Suppose $g : W \to X, f : X \to Y$ and a submanifold $Z \subset Y$ are given, with $f \pitchfork Z$ on X. Show that if $f \circ g$ and Z are appropriate for intersection theory, so are g and $f^{-1}(Z)$, and prove that:

$$I(f \circ g, Z) = I(g, f^{-1}(Z)).$$

6. (i) Prove that the Euler characteristic of the product of two compact, oriented manifolds is the product of their Euler characteristics.

(ii) Let Z be a compact submanifold of Y, both oriented, with $dim(Z) = \frac{1}{2}dim(Y)$. Prove that:

$$I(Z, Z) = I(Z \times Z, \Delta).$$

(See the hint in [G-P, p.118].) Δ is the diagonal submanifold of $Y \times Y$.