EXISTENCE OF TUBULAR NEIGHBORHOODS

Let $M^m \subset R^{m+n}$ be a submanifold. At each $p \in M$, the tangent space T_pM is a subspace of R^{m+n}, and we denote by $\nu_p M$ (normal space at p) its orthogonal complement. These subspaces fit together to give a vector bundle over M, which can be given a C^{k-1} manifold structure if M is of class C^k ($k \geq 2$):

$$\nu M = \{(p, v) \in M \times R^{m+n}; v \in \nu_p M\},$$

the normal bundle of M in R^{n+m}, a vector bundle of rank n (fiber dimension) over M. We also consider the open normal ball of radius $\epsilon > 0$ at p:

$$B^{\perp}(p, \epsilon) = \{p + v \in R^{m+n}; v \in \nu_p M, |v| < \epsilon\},$$

a union of normal segments $[p, p+v], |v| < 1$, based at p, normal to M.

Example: *Preimages of regular values.* If $f : U \to R^n$ is a C^r map ($U \subset R^{m+n}$ open) , $c \in R^n$ is a regular value and $M = f^{-1}(c)$, the gradients ∇f^i of the components of f give n linearly independent normal vector fields, globally defined on M and linearly independent at each point. (They are l.i. since c is a regular value, so at any point $p \in M$, the differential $df(p) \in \mathcal{L}(R^{m+n}, R^n)$ is surjective, and its rows are the ∇f^i.) Thus the normal bundle νM is 'trivial' in this case, that is, equivalent to the product bundle $M \times R^n$.

Note that, by the definition of submanifold, this is always true locally: given any $p \in M$, there exists a neighborhood $U \subset R^{n+m}$ (open) of p, and a map $f : U \to R^n$ so that $0 \in R^n$ is a regular value and $M \cap U = f^{-1}(0)$, and thus we have n linearly independent normal vector fields defined in $M \cap U$.

An *admissible normal radius* for M is a number $\epsilon_M > 0$ so that normal balls based at different points of M, with radius less than or equal to ϵ_M, do not intersect.

Theorem 1: *Tubular neighborhood of a compact submanifold.* Let $M^m \subset R^{m+n}$ be a compact submanifold (of class C^k). Then there exists an admissible normal radius ϵ_M for M. If $\epsilon \leq \epsilon_M$, the set:

$$V_\epsilon(M) = \bigsqcup_{p \in M} B^{\perp}(p, \epsilon)$$

(disjoint union) is open in R^{m+n}. The nearest-point retraction $\pi : V_\epsilon(M) \to M$, defined by taking each $x \in V_\epsilon$ to the center p of the unique normal ball $B^{\perp}(p, \epsilon)$ containing x, is a C^{k-1} map.

Proof. (i) The theorem is true locally: any $p \in M$ has an open neighborhood $U \subset M$ for which an admissible normal radius ϵ_U exists. To see this, consider local parameters at p, $\phi : V_0 \to V$ (diffeo of class C^k, $V_0 \subset R^m$ open, V an open neighborhood of p in M). On V we have n linearly independent vector fields $v_1, \ldots, v_n$ (of class C^{k-1}), which we may assume orthonormal. Define the C^{k-1}

map $\Phi : V_0 \times R^n \to R^{m+n}$ by:

$$\Phi(x, t^1, \dots, t^n) = \phi(x) + \sum_i t^i v_i(\phi(x))$$

(addition in R^{n+m}). The Jacobian $[d\Phi]_|(x,0)$ at a point of $V_0 \times \{0\} \subset R^{n+m}$ has as its first m columns the vectors $\partial_{x_i}\phi(x)$, as last n the vectors $v_1(x), \dots, v_n(x)$. Since the first m vectors are a basis for $T_{\phi(x)}M$, the last n a basis for $\nu_{\phi(x)}M$, it follows $d\Phi(x,0)$ is an isomorphism.

By the inverse function theorem, if $\phi(x_0) = p$ there exists an open neighborhood $U_0 \times \mathbb{B}_\epsilon$ of $(x_0, 0)$ in $R^m \times R^n$, on which Φ is a diffeomorphism to an open set in R^{n+m} of the form:

$$V_\epsilon(U) = \bigsqcup_{q \in U} B^\perp(q, \epsilon), \quad U = \phi(U_0) \subset M.$$

It's the same ϵ, since the fact the normal frame $\{v_i\}$ in V is orthonormal implies $\Phi(x, \cdot)$ is an isometry from $\mathbb{B}_\epsilon$ in R^n to $B^\perp(\phi(x), \epsilon)$, for each $x \in U_0$. And the union is disjoint, since the normal balls are images under the diffeo. Φ of disjoint sets $\{x\} \times \mathbb{B}_\epsilon, \{x'\}\times, \mathbb{B}_\epsilon$, for $x \neq x'$ in U_0.

Within the open set $V_\epsilon(U)$ (which contains U), the nearest-point map $\pi(q, v) = q \in U$, $v \in B^\perp(q, \epsilon)$, is C^{k-1}, since it is the inverse of Φ, followed by projection from $U_0 \times \mathbb{B}_\epsilon$ to the first factor U_0, and then followed by the C^k diffeo. $\phi : U_0 \to U$.

(ii) The theorem holds globally on M: from compactness, we may cover M by finitely many coordinate neighborhoods $U_1, \dots, U_r$, in each of which the conclusions of part (i) of the proof holds. So we have admissible normal radii $\epsilon_1, \dots, \epsilon_r$. This finite open cover also has a Lebesgue number λ, so let:

$$\epsilon_M = \min\{\frac{\lambda}{2}, \epsilon_1, \dots, \epsilon_r\}.$$

Then if $\epsilon < \epsilon_M$, two normal segments of length ϵ and different basepoints $p, q \in M$ may not intersect. This is clear if p, q are in the same U_i. If they're in different sets of the cover, the distance $|p - q| \geq \lambda > 2\epsilon$; so p, q can't be basepoints of intersecting euclidean segments of length ϵ. Clearly we have:

$$V_\epsilon(M) = \bigcup_{i=1}^r V_\epsilon(U_i),$$

and therefore is an open set. That the nearest-point retraction $\pi : V_\epsilon(M) \to M$ is a C^{k-1} map is a local statement, hence already shown in part(i).

We record the case of trivial normal bundle, Say a tubular neighborhood $V_\epsilon(M)$ is *equivalent to a product* if there exists a diiffeomorphism $h : M \times \mathbb{B}_\epsilon \to V_\epsilon(M)$ which preserves fibers (takes $\{x\} \times \mathbb{B}_\epsilon$ to $B^\perp(p, \epsilon)$, $p = h(x, 0)$).

Proposition. For a submanifold $M^m \subset R^{m+n}$ (compact or not), the following are equivalent:

1) M is the preimage of a regular value $a \in R^n$, for some smooth map $f : U \to R^n$, $U \subset R^{m+n}$ open.

2) M admits n linearly. independent, globally defined *normal* vector fields, $v_i : M \to R^{m+n}, v_i(p) \in \nu_p M$.

3) Any tubular neighborhood of M in R^{m+n} is equivalent to a product.

Exercise. Show that the graph (in R^2) of the curve $y = x^{4/3}$, $x \in R$ (a submanifold of R^2 that is C^1, but not C^2), does not have an admissible normal radius.

Case of non-compact submanifolds of R^{n+m}.

Simple examples show that even properly embedded, smooth noncompact manifolds may fail to have a globally admissible normal radius. But it does admit a normal radius that is a *function* on M.

Theorem 2. Let $M^m \subset R^{m+n}$ be an embedded submanifold, compact or not. There exists a positive continuous function $\epsilon : M \to R$ so that:

1) $p \neq q$ on M $\Rightarrow B^\perp(p, \epsilon(p)) \cap B^\perp(q, \epsilon(q)) = \emptyset$;

2) The set $V_\epsilon(M) = \bigsqcup_{p \in M} B^\perp(p, \epsilon(p))$ (disjoint union of normal balls) is an open neighborhood of M in R^{m+n};

3) The nearest point retraction $\pi : V_\epsilon(M) \to M$ is well-defined and smooth (if M is smooth; C^{k-1} if M is C^k.)

Lemma. Given any compact subset $K \subset M$, we may find an admissible normal radius $\alpha_K > 0$ with the property:

$$\overline{V_{\alpha_K}(K)} \cap M = K, \quad V_{\alpha_K}(K) = \bigsqcup_{p \in K} B^\perp(p, \alpha_K).$$

(The point is the admissible radius can be taken small enough that the tubular nbd. of K with that radius doesn't 'bump up' against other parts of M, far from K.)

Proof of Lemma. Let $L \subset M$ be a compact neighborhood of K, let α_L be an admissible normal radius for L, and set:

$$\alpha_K = \frac{1}{2} \min\{\alpha_L, d(K, M \setminus L)\}.$$

Let $q \in \overline{V_{\alpha_K}(K)} \cap M$. Then since $d(q, K) \leq \alpha_L < d(K, M \setminus L)$. we have $q \in L$. And also q is in a normal segment to M of length $< \alpha_L$, based at $p \in K \subset L$. Thus necessarily $p = q$, so $q \in K$.

Proof of theorem. Consider a compact exhaustion of M, $M = \bigcup_{i \geq 1} K_i, K_i \subset int(K_{i+1})$. Let α_i be the admissible normal radius associated to K_i by the lemma, where we assume $\alpha_{i+1} \leq \alpha_i$. Now define a new sequence of admissible normal radii, starting with $\epsilon_1 = \alpha_2, \epsilon_2 = \alpha_3$.

Inductively, assuming $\epsilon_1 \geq \epsilon_2 \geq \epsilon_3 \geq \ldots \geq \epsilon_s$ defined, choose ϵ_{s+1} by requiring $\epsilon_{s+1} \leq \min\{\epsilon_s, \alpha_{s+2}\}$ and:

$$\epsilon_{s+1} \leq dist(K_{s+1} - int(K_s), \bigcup_{i=1}^{s-1} V_{\epsilon_i}(K_i)).$$

We *claim* this choice accomplishes the following:

$$\text{If } p \in K_i, q \in K_j, p \neq q, \text{ then } B^{\perp}(p, \epsilon_i) \cap B^{\perp}(q, \epsilon_j) = \emptyset.$$

Indeed if $i, j \leq s$, this is the induction hypothesis. If $i = s, j = s+1$, this is true since $\epsilon_s \leq \alpha_{s+1}$. The remaining case is:

$$p \in K_s \setminus int(K_{s+1}) \text{ and } q \in K_{i_0}, \text{ with } i_0 < s.$$

Then we consider two normal segments to M, $[p, a]$ with length $< \epsilon_{s+1}$, $[q, b]$ with length $< \epsilon_{i_0}$. We have:

$$[q, b] \subset V_{\epsilon_{i_0}}(K_{i_0}) \subset \bigcup_{i=1}^{s-1} V_{\epsilon_i}(K_i),$$

and then the second condition used above in the choice of ϵ_{s+1} implies $\epsilon_{s+1} < dist(p, [q, b])$. That is, the length of $[p, a]$ is smaller than the distance from p to $[q, b]$. Thus $[p, a] \cap [q, b] = \emptyset$, proving the claim.

To prove the theorem, let $V(M) = \bigcup_{i=1}^{\infty} V_{\epsilon_i}(K_i)$, and define $\epsilon : M \to R_+$ as:

$$\epsilon(p) = dist(p, R^{m+n} \setminus V(M)).$$

This is a positive continuous function on M, and $0 < \epsilon(p) \leq \epsilon_i$, if $p \in K_i \setminus K_{i-1}$. So $V_\epsilon(M) \subset V(M)$, and any $x \in V_\epsilon(M)$ lies in a unique segment normal to M. Thus the nearest-point retraction $\pi : V_\epsilon(M)$ is well-defined.

To see that $V_\epsilon(M)$ is open, proceed as in the compact case and cover M by domains U of charts, with normal vector fields $v_1, \ldots, v_n$ defined and orthonormal in U, and local parameters $\phi : U_0 \to U$ ($U_0 \subset R^m$ open) so that with A the open set: in $R^m \times R^n$

$$A = \{(x, y) \in U \times R^n; |y| < \epsilon(\phi(x))\}$$

we have a diffeomorphism $\Phi : A \to \pi^{-1}(U)$, $\Phi(x, y) = \phi(x) + \sum_i y^i v_i(\phi(x))$. So $\pi^{-1}(U)$ is open in R^{m+n}, and $V_\epsilon(M)$, as the union of the $\pi^{-1}(U)$ over all such U in the cover, is open too. The fact that π is of class C^{k-1} is a local statement, proved exactly as in the compact case. This concludes the proof of the theorem.

For submanifolds $M^m \subset V^{m+n}$ of (smooth) manifolds, we have the following: define a *tubular neighborhood* of M in V to be a smooth embedding $\Phi : N \to V$ of a rank n vector bundle $p : N \to M$ into V, with image an open neighborhood $\mathcal{T}$ of M in V, so that:

(i) A smooth retraction $\pi : \mathcal{T} \to M$ is defined, and $\pi \circ \Phi = p$;

(ii) The embedding maps the zero section to M, as the identity ($\Phi(x, 0) = x, \forall x \in M$), and is transversal on fibers:

$$d\Phi_{|(x,0)}[N_x] \oplus T_x M = T_x V, \quad x \in M.$$

Theorem 3. Any submanifold $M \subset V$ admits a tubular neighborhood $\mathcal{T}$, as defined above.

For the proof, see [Hirsch, thm 5.2, p.110].