MATH 562, Hw Set 1 (spring 2023): partial solutions

1. *Claim:* Any closed interval in S^1 is a retract of S^1.

Let $I \subset S^1$ be a closed interval defined by a chord C, and let $\bar{I}$ be the complementary closed interval (so I and $\bar{I}$ intersect at their endpoints; we ignore the trivial cases: a single point and all of S^1.) There are three cases to consider:

(i) $length(I) = \pi$. Then C is a diameter; lett $\rho_C : S^1 \to S^1$ denote the reflection on C, we define the retraction $r : S^1 \to I$ setting $r(x) = x$ for $x \in I$, $r(x) = \rho_C(x) \in I$, for $x \in \bar{I}$.

(ii) $length(I) < \pi$. Then there exists a interval J of length π containing I. Let $\rho_{IJ} : J \to I$ be the retraction which fixes I pointwise, and maps each of the two intervals in $J \setminus I$ to the closest endpoint of I. Then if $r_J : S^1 \to J$ is the retraction described in (i), the map $r = \rho_{IJ} \circ r_J : S^1 \to I$ is a retraction.

(iii) $length(I) > \pi$. Then $length(\bar{I}) < \pi$, and let $I' = \rho_C(\bar{I})$, the image of $\bar{I}$ under the reflection on the chord C defining I. Let $q \in S^1$ be the closest point to C on the arc $\bar{I}$ (so q is the endpoint on $\bar{I}$ of the diameter perpendicular to C), and denote by $\rho_q : I' \to I$ the radial projection from q (which in fact defines a homeomorphism from I' to I). Then the map $r : S^1 \to I$ defined as $r(x) = x, x \in I$; $r(x) = \rho_q(\rho_C(x)), x \in \bar{I}$ is continuous, and a retraction onto I.

2. The Gram-Schmidt process defines a deformation retraction of matrix groups (with real entries):

$$O_n \subset GL_n.$$

By restriction, we obtain that each successive inclusion below is a deformation retraction:

$$SO_n \subset SL_n \subset GL_n^+$$

(the last group are the $n \times n$ matrices with positive determinant.)

Solution. Gram-Schmidt constructs, from a given basis $\{v_1, \ldots, v_n\}$ of R^n, an orthonormal basis $\{e_1, \ldots, e_n\}$, in such a way that:

$$\{e_1, \ldots, e_i\} \in span_R\{v_1, \ldots, v_i\}, \text{ for each } i = 1, \ldots, n.$$

In matrix terms, this says that, letting $V = [v_1|...|v_n]$, $U = [e_1|...|e_n]$ (by columns), we have:

$$U = VT,$$

where T is an upper-triangular $n \times n$ matrix (zeros below the diagonal), with positive diagonal entries.

So let $B_n \subset GL_n$ be the group of upper-triangular matrices with positive diagonal entries. B_n is contractible, since the deformation:

$$T_s = s\mathbb{I}_n + (1-s)T \in B_n, \quad T \in B_n, s \in [0,1],$$

defines a homotopy in B_n from the identity map in B_n to the constant map $\mathbb{I}_n \in B_n$. Now, Gram-Schmidt defines a continuous map:

$$\mathcal{G} : GL_n \to B_n,$$

so that for each $V \in GL_n$:

$$V\mathcal{G}(V) \in O_n.$$

Consider then the deformation of a given $V \in GL_n$:

$$V_s = V\mathcal{G}_s(V) \in GL_n, \quad \mathcal{G}_s(V) = [(1-s)\mathbb{I}_n + s\mathcal{G}(V)] \in B_n, \quad s \in [0,1].$$

We see that $V_0 = V, V_1 \in O_n$. Thus the map $V \mapsto V\mathcal{G}(V)$ defines a deformation retraction from GL_n to O_n.

Now, since $det(V\mathcal{G}_s(V)) > 0$ if $det(V) > 0$ (since the diagonal entries of $\mathcal{G}_s(V) \in B_n$ are positive), we see that $V_s \in GL_n^+$ if $V \in GL_n^+$, so by restriction (bearing in mind $SO_n = O_n \cap GL_n^+$) we obtain a deformation retraction $V \mapsto V\mathcal{G}(V)$ from GL_n^+ to SO_n. And the restriction of this map to SL_n defines a deformation retraction from SL_n to $SO_n = O_n \cap SL_n$.

That SL_n is a deformation retract of GL_n^+ can be seen directly. The deformation:

$$V_s = [(1-s)+s(det(V))^{-\frac{1}{n}}]V, \quad s \in [0,1]; \quad V_s \in GL_n^+ \text{ if } V_0 = V \in GL_n^+, V_1 \in SL_n,$$

establishes a deformation retraction from GL_n^+ to SL_n.