MATH 562, SPRING 2023–FINAL EXAM (May 15, 2023)– Solve EIGHT of the twelve problems proposed. (Time given: 150 minutes.)

1. (i) X Hausdorff, $Y \subset X$ subspace. Prove that Y is a retract of X if, and only if, for any space Z, any continuous $f : Y \to Z$ admits a continuous extension $\bar{f} : X \to Z$.

(ii) Let $E = \{(0, 0, z) \in R^3; z \in R\}$. Show that $R^3 \setminus E$ has the homotopy type of S^1.

2. State the Borsuk-Ulam theorem for maps $f : S^k \to R^{k+1} \setminus \{0\}$, then use it to prove that if $g : S^k \to S^k$ satisfies $g \circ \alpha = \alpha \circ g$ (where α is the antipodal map of S^k) then $deg_2 g = 1$.

3. Let X, Y be submanifolds of R^N. Show that for almost every $a \in R^N$, the translate $X + a$ intersects Y transversely.(State precisely any transversality theorem you use for this, and verify its hypotheses hold.)

4. Let $p : S^1 \to S^1$, $p(z) = z^2$ $(z \in \mathbb{C}, |z| = 1)$.

(i) Prove p is a covering map, and that the covering is regular;

(ii) With $x_0 = \tilde{x}_0 = 1 \in \mathbb{C}$, identify (with proof) the subgroup $H(\tilde{x}_0) = p_*\pi_1(S^1, \tilde{x}_0)$ of $\pi_1(S^1, x_0)$, and the automorphism group of the covering.

5. (i) Define "properly discontinuous group action" (of a group G of homeomorphisms on a space Y).

(ii) Let $p : \tilde{X} \to X$ be a covering map. Prove that the automorphism group $G = Aut(\tilde{X}|X)$ of the covering acts properly discontinuously on $\tilde{X}$.

6. Let $\alpha : S^k \to S^k$ be the antipodal map, $\alpha(x) = -x$. Show that if S^k admits a nonvanishing vector field, then α is homotopic to the identity, and that this implies k is odd.

7. Let $f : R^k \to R^k$ smooth, $0 \in R^k$ a regular point of f(with $f(0) = 0$), $g : \partial B \to R^k \setminus \{0\}$ the restriction of f to the boundary of a small ball B at the origin, containing no other points of $f^{-1}(0)$. Prove the winding number $W(g, 0)$ equals ± 1, depending on whether f preserves or reverses orientation at 0.

Hint: We have $f(x) = df_0[x] + r(x)$ with $r(x)/|x| \to 0$ as $x \to 0$. Define a homotopy between $f/|f|$ and $df_0/|df_0|$ on ∂B, then recall the group GL_k^+ of $k \times k$ matrices with positive determinant is connected.

8. (i) Suppose V is a vector field with isolated zeros in R^k, and W is a compact k-dimensional submanifold with boundary of R^k; assume V is never zero on ∂W. Prove that the sum of the indices of V at its zeros inside W equals the degree of the map:

$$\frac{V}{|V|} : \partial W \to S^{k-1} \quad (k \geq 2).$$

(ii) Define 'Lefschetz map' (of a compact oriented manifold without boundary); prove that the local Lefschetz number at the origin of the map of R^k $f(x) = \frac{1}{2}x$ is $L_0(f) = (-1)^k$.