MATH 562, SPRING 2023–FINAL EXAM-SOLUTIONS

1. (i) If every $f : Y \to Z$ extends (for every Z), in particular the identity map of Y, $id_Y : Y \to Y$ extends to $g : X \to Y$ continuous; by definition, g retracts X to Y. Conversely, if a retraction $g : X \to Y$ exists (so $g_{|Y} = id_Y$), we may extend an arbitrary $f : Y \to Z$ by setting $\bar{f} = f \circ g : X \to Z$.

(ii) We need $f : E^c \to S^1$ and $g : S^1 \to E^c$ so that $f \circ g \simeq id_{S^1}$ and $g \circ f \simeq id_{E^c}$. Let g be the inclusion map; then $f \circ g = id_{S^1}$ is guaranteed if f is a retraction; and $g \circ f \simeq id_{E^c}$ is guaranteed if f is a deformation retraction, that is, if f may be joined to id_{E^c} by a continuous path $f^t : E^c \to E^c$.

Write $f = f_2 \circ f_1$, where $f_1 : E^c \to R^2 \setminus 0$ is vertical projection $f_1(x, y, z) = (x, y, 0)$ and $f_2 : R^2 \setminus 0 \to S^1$ is the radial projection. Both are deformation retractions (via $f_1^t(x, y, z) = (x, y, (1-t)z)$ and $f_2^t(v) = (1-t)v + t\frac{v}{|v|}$ respectively), and therefore so is f. $\square$.

2. Borsuk-Ulam theorem: if $f : S^k \to R^{k+1} \setminus 0$ is an odd map ($f(-x) = -f(x)$), then the mod 2 winding number $W_2(f, 0)$ equals 1 (mod 2). A map $g : S^k \to S^k$ commutes with α exactly if it is odd as a map to $R^{k+1} \setminus 0$; and then, by definition of the winding number, $W_2(g, 0) = deg_2 g = 1$ (mod 2).

3. Define $f : X \times R^N \to R^N$ by $f(x, a) = x + a$. Then the partial differential with respect to the second argument $d_2 f(x, a) \in \mathcal{L}(R^N)$ is the identity for all (x, a); in particular f is a submersion, and transversal to any submanifold $Y \subset R^N$. By the (parametrized) transversality theorem, for almost every $a \in R^N$ the map $f_a : X \to R^N$, $f_a(x) = x + a$, will be transversal to Y; so either $im(f_a) \cap Y = \emptyset$ (the translate of X and Y do not intersect at all), or they intersect transversely (possible only if $dim(X) + dim(Y) \geq N$).

4. (i) Since $p'(z) = 2z \neq 0$ for $z \neq 0$, the map p is a local diffeomorphism as a map from $R^2 \setminus 0$ to itself, hence restricts to a surjective local diffeomorphism of S^1. Since S^1 is compact, p is a covering map. (And clearly a 2-1 cover, since each point on S^1 has two preimages on S^1.) Since the fundamental group of S^1 is abelian, the covering is regular.

(ii) Let $\gamma(t) = e^{2\pi i t}, t \in [0, 1]$, represent a generator $u = [\gamma]$ of $\pi_1(S^1, 1) = \mathbb{Z}$. Then $f \circ \gamma(t) = e^{4\pi i t}$, so $f_* u = 2u$ and $H(1) = 2\mathbb{Z}$ (even integers), while the automorphism group of the cover is isomorphic to $\pi_1(S^1, 1)/H(1) \approx \mathbb{Z}_2$.

5. (i) G acts properly discontinuously on Y if $\forall y \in Y$ $\exists U \subset Y$ nbd. of y, so that $gU \cap U = \emptyset, \forall g \in G$, $g \neq id_Y$.

(ii) Let $x \in \tilde{X}, p(x) = y \in X$, $V \subset X$ be an evenly covered neighborhood of Y, so $p^{-1}(V) = \bigsqcup_{\lambda \in \Lambda} U_\lambda$, all disjoint. Say $x \in U_{\lambda_0}$. Then if $g \in Aut(\tilde{X}|X), g \neq id$, we have $g(U_{\lambda_0}) = U_\lambda$ for some $\lambda \neq \lambda_0$, and therefore $g(U_{\lambda_0}) \cap U_{\lambda_0} = \emptyset$.

6. Suppose S^k admits a nonvanishing vector field V; to see α is homotopic to the identity (and thus $deg(\alpha) = 1$, so k is odd), consider the map of S^k:

$$f(x) = \frac{x + V(x)}{|x + V(x)|}.$$

This map is well-defined ($V(x)+x = 0$ is not possible, since $\langle V(x), x\rangle = 0$) and homotopic to the identity on S^k (replace V by tV, $\in [0,1]$.) But f doesn't have any fixed points on S^k (since $V(x) \neq 0$ for all x), and hence is homotopic to α via the normalized linear homotopy:

$$f_t(x) = \frac{t\alpha(x) + (1-t)f(x)}{|t\alpha(x) + (1-t)f(x)|}, \quad t \in [0,1].$$

7. Let $A = df_0 \in Iso(R^k)$ (since $0 \in R^k$ is a regular value and $f(0) = 0$). Let B be a closed ball at the origin, with small radius; say $|Ax| > c|x|$ for $x \in R^k$ (with $c > 0$), and B has small enough radius that $|r(x)| < \frac{c}{2}|x|$ for $x \in \partial B$. Then since $f(x) = Ax + r(x)$, we have $|f(x)| > c|x| - \frac{c}{2}|x| > 0$ for $x \in \partial B$. The winding number $W(g, 0)$ equals (by definition) the degree of $\frac{g}{|g|} : \partial B \to S^{k-1}$. But $\frac{g}{|g|}$ is homotopic to $\hat{A}(x) = \frac{Ax}{|Ax|}$ on ∂B, via:

$$h^t(x) = \frac{Ax + (1-t)r(x)}{|Ax + (1-t)tr(x)|}, \quad h^t : \partial B \to S^{k-1}, \quad h^1(x) = \hat{A}(x), h^0(x) = \frac{g(x)}{|g(x)|}, \quad t \in [0,1],$$

where we already established the denominator does not vanish on ∂B. By homotopy invariance, the degree of $g/|g|$ on ∂B equals that of $\hat{A}$. And $\hat{A}$ is either homotopic (as a diffeomorphism of the sphere) either to the identity (if $det(A) = 1$) or to a reflection (if $det(A) = -1$), so its degree is ± 1.

8. (i) V has finitely mani zeros $x_1, \ldots, x_N$ in $int(W)$, since W is compact and the zeros are isolated. Let B_i be a small ball centered at x_i, contained in $int(W)$ and containing no other zeros of V in its closure. Let $W_1 = W \setminus \bigsqcup_{i=1}^N B_i$, and set $\hat{V} = \frac{V}{|V|}$ (taking values in S^{k-1}). Then $deg(\hat{V}; \partial W_1) = 0$, since $\hat{V}$ is defined in W_1. Since the oriented boundary of W_1 is $\partial W - \sum_i \partial B_i$, we have $deg(\hat{V}; \partial W) = \sum_i \deg(\hat{V}; \partial B_i) = \sum_i ind(V; x_i)$, by definition of the index of V at x_i.

(ii) A smooth map $f : X \to X$ (X a compact oriented manifold without boundary) is a *Lefschetz map* if $Graph(f) \pitchfork \Delta_X$, as submanifolds of $X \times X$ (Δ_X is the diagonal submanifold). For a linear Lefschetz map $f : R^k \to R^k$ (that is, 1 is not an eignevalue) the local Lefschetz number is the sign of the determinant of $f - Id_{R^k}$. In this case, $L_0(f) = sign(\frac{1}{2} - 1)^k = (-1)^k$.