MATH 562, SPRING 2023/ HOMEWORK SET 3 (Problems from [Lima].)

1. (p.50) Let M be a properly embedded n-dimensional smooth submanifold of R^k. If $k \geq n+2$, prove that $R^k \setminus M$ is connected. If $k \geq n+3$, $R^k \setminus M$ is simply connected.

2. (p. 50) Let $X = \bigcup_{n \geq 1} U_n$ be an increasing union of open subsets ($U_n \subset U_{n+1}$). If each U_n is simply-connected, then X is simply-connected.

3. Problem 13, p.50: prove that the subspace $Z \subset R^2$ pictured is simply-connected, but not locally connected.

4. (p.179) Let X be the figure-eight space, and $\tilde{X}$ be the subset of the upper-half plane consisting of the x-axis, and all circles of radius 1/3 tangent to the x-axis at points $(n, 0), n \in \mathbb{Z}$. Define a covering map p from $\tilde{X}$ onto X, and determine if this covering is regular or not. With $\tilde{x}_0 = (0, 0)$, find the subgroup $H(\tilde{x}_0)$ of $\pi_1(X, x_0), x_0 = p(\tilde{x}_0)$. Determine also the group of covering automorphisms $Aut(\tilde{X}|X)$.

5. (p. 149) Let $p : \tilde{X} \to X$ be a covering, with $\tilde{X}$ connected and $p^{-1}(x)$ finite, for all $x \in X$. If there exists a continuous function $f : \tilde{X} \to \mathbb{R}$, injective on each fiber $p^{-1}(x)$, then p is a homeomorphism.

6. (p.72) Let $f : S^1 \to S^1$ be an odd map ($f(-x) = -f(x)$). Then the degree of f is odd.

7. (p.114) Let $f : U \to \mathbb{C} \setminus \{0\}$ be continuous, $U \subset \mathbb{C}$ open. There exists $g : U \to \mathbb{C}$ continuous, such that $f(z) = e^{g(z)}$ for all $z \in U$ if, and only if, for each closed path c in U the degree (with respect to the origin) of the path $f \circ c$ is zero.

The following problem will be discussed later, and is not included in the homework set.

8. (Not in [Lima]) (i) The fundamental group of the Hawaiian earrings is not countable.

(ii) Prove the Hawaiian earrings space *is not* homeomorphic to a countable wedge of circles (the quotient space of a countable union of circles S_i with points $x_i \in S_i$ identified to a single point.)

Hint: See [Munkres, p.436] for ideas.