MATH 562, SPRING 2023–HOMEWORK 5 SOLUTIONS

1. $S^k = g^{-1}(1)$, and the normal line bundle $\{N_x\}_{x\in S^k}$ of S^k in R^{k+1} is oriented by the requirement $dg_x[n_x] > 0$. Since $dg_x[v] = 2\langle x, v\rangle$ for $v \in R^{n+1}$, it follows we must have $n_x = cx$ with $c > 0$, the outward normal. Then the preimage orientation is defined by requiring the splitting $R^{k+1} = N_x \oplus T_x S^k$ to be oriented; equivalently, by declaring a basis $\{v_1, \ldots, v_k\}$ of $T_x S^k$ to be. positive if $\{n_x, v_1, \ldots, v_k\}$ is a positive basis of R^{k+1}. Since n_x is the outward normal, this is the same as the orientation induced on S^k as the boundary of the open submanifold $B^{k+1} \subset R^{k+1}$.

2. Here the graph $S = F^{-1}(0)$, and again the normal line bundle $N = \{N_p\}_{p\in S}$ is oriented by requiring $dF_p[n_p] > 0$. Since $dF_p[v] = v_3 - f_x v_1 - f_y v_2$ for $v \in R^3$ and $p = (x, y, f(x, y)) \in S$, setting $n_p = (-f_x, -f_y, 1)$ we find $dF_x[n_x] = f_x^2 + f_y^2 + 1 > 0$, so we orient N by n_p, the upward normal to the graph at p. The preimage orientation on S is then defined by declaring a basis $\{v_1, v_2\}$ of $T_p S$ to be positive ($p \in S$) if $\{n_p, v_1, v_2\}$ is a positive basis of R^3. For the choice $v_1 = (1, 0, f_x), v_2 = (0, 1, f_y)$, we see that $det[n_p|v_1|v_2] = f_x^2 + f_y^2 + 1 > 0$, so this is a positive basis of $T_p S$.

3. By the Jordan-Brouwer theorem, $M^n = \partial D$, where $D \subset R^{n+1}$ is the bounded component of the two into which M separates R^{n+1}. As the boundary of the manifold with boundary D, M is oriented by the unit outward normal, defined as the unit normal pointing into the unbounded component of the complement, at each point of M.

4. (i) $\alpha : S^k \to S^k$ preserves orientation for k odd, reverses for k even, and is a diffeomorphism (so each point on S^k has a unique preimage). Thus $deg(\alpha) = (-1)^{k+1}$.

If α is homotopic to the identity, $deg(\alpha) = 1$, so k is odd. Conversely, if k is odd, $S^k \subset R^{k+1}$ has a nonvanishing tangent vector field:

$$V(x) = (x_2, -x_1, \ldots, x_{2n}, -x_{2n-1}), \quad k + 1 = 2n.$$

And if S^k admits a nonvanishing vector field V, it follows that α is homotopic to the identity (and thus k is odd). To see this, consider the map of S^k:

$$f(x) = \frac{x + V(x)}{||x + V(x)||}.$$

This map is well-defined ($V(x) + x = 0$ is not possible, since $\langle V(x), x\rangle = 0$) and homotopic to the identity on S^k (replace V by tV, $\in [0, 1]$.) But f doesn't have any fixed points on S^k (since $V(x) \neq 0$ for all x), and hence is homotopic to α via the normalized linear homotopy:

$$f_t(x) = \frac{t\alpha(x) + (1 - t)f(x)}{||t\alpha(x) + (1 - t)f(x)||}, \quad t \in [0, 1].$$

This proves points (ii) and (iii).

5. (i) $dim(X) = dim(Y) = dim(Z)$, with X, Y, Z oriented and compact. In this situation, z is a regular value for g iff, for each $y \in g^{-1}(z)$, we may find neighborhoods $V \subset Y$ of y and $W \subset Z$ of w so that $g_{|V} : V \to W$ is a diffeomorphism; and similarly for regular values of f. Then Sard's theorem and a simple continuity argument show one may choose a regular value z for g so that its preimage $g^{-1}(z)$ consists of regular values for f. And then we have:

$$(g \circ f)^{-1}(z) = f^{-1}(g^{-1}(z)) = \bigsqcup_{y \in g^{-1}(z)} f^{-1}(y),$$

a finite disjoint union of finite sets.

The degree of the composition is:

$$deg(g \circ f) = \sum_{y \in g^{-1}(z)} \sum_{x \in f^{-1}(y)} sign(dg_y \circ df_x),$$

and since $sign(dg_y \circ df_x) = sign(dg_y) sign(df_x)$, this equals:

$$deg(g \circ f) = [\sum_{y \in g^{-1}(z)} sign(dg_y)][\sum_{x \in f^{-1}(y)} sign(df_x)] = deg(g) deg(f).$$

(ii) We have $f : X \to Y$ and $g : W \to X$, with W compact, and $Z \subset Y$, all with empty boundary. The expression "$f \circ g$ and Z are appropriate for intersection theory" means W, Z and Y are oriented and $dim(W) + dim(Z) = dim(Y)$. Here it is assumed that $f \pitchfork Z$, so $S = f^{-1}(Z)$ is a submanifold of X, with $codim_X S = codim_Y Z$, which also equals $dim(W)$. Thus $dim(W) + dim(S) = dim(X)$. Since S can be given the preimage orientation (defined by f and by the given orientations of X, Y and Z), we also have "g and S are appropriate for intersection theory".

Next we must show $I(g, S) = I(g \circ f, Z)$. Replacing g by a homotopic map (which doesn't change the intersection numbers), we may assume g is transversal to S; by the dimension and compactness conditions, it follows that $g^{-1}(S) = (f \circ g)^{-1}(Z)$ is a finite subset of W. Fix an arbitrary w in this finite set, and let $s = g(w) \in S$. Let $E_s = dg_w[T_w W] \subset T_s X$. By transversality of g to S (at w) and dimension count, we see that:

$$E_s \oplus T_s S = T_s X, \tag{1}$$

and that dg_w is an isomorphism from $T_w W$ to E_s. From transversality of f to Z (at $s \in S$), we also know:

$$df_s[T_s X] + T_{f(s)} Z = T_{f(s)} Y \tag{2}$$

(not a direct sum, until proof to the contrary). To establish the equality of intersection numbers, we must show:

$$df_s[E_s] \oplus T_{f(s)} Z = T_{f(s)} Y \tag{3}$$

(Note $df_s[E_s] = d(f \circ g)_w[T_w W]$, and $f(s) = (f \circ g)(w)$.) Since $df_s[T_s S] \subset T_{f(s)}Z$, it follows from (1) and (2) that we do have:

$$df_s[E_s] + T_{f(s)}Z = T_{f(s)}Y,$$

and we must show this is a direct sum. This last decomposition already implies $dim(df_s[E_s]) \geq dim(Y) - dim(Z)$, the first in the chain of inequalities:

$$dim(Y) - dim(Z) \leq dim(df_s[E_s]) \leq dim(E_s) = dim(X) - dim(S).$$

Since the rightmost and leftmost numbers coincide, we must have:

$$dim(df_s[E_s]) = dim(E_s) = dim(Y) - dim(Z),$$

implying (3).

The point $w \in f^{-1}(S)$ is positive for the intersection number of $f \circ g$ with Z if (3) is an oriented direct sum, and this is the case if and only if the direct sum in (1) is also oriented, where S is given the preimage orientation. (This is exactly the definition of 'preimage orientation', given that df_s is an isomorphism on E_s.) Since $w \in g^{-1}(S) = (f \circ g)^{-1}(Z)$ is arbitrary, this concludes the proof that the intersection numbers of $f \circ g$ with Z and of g with S are equal.

6. (i) Let M, N be compact and oriented; take the product orientation on $M \times N$. Let $f : M \to M$, $g : N \to N$ be homotopic to id_M, id_N (resp.) and with graphs Γ_f, Γ_g transversal to the diagonals Δ_M, Δ_N in $M \times M, N \times N$ (resp); that is, f and g are Lefschetz maps.Then $\chi(M) = \#Fix(f)$, $\chi(N) = \#Fix(g)$ (finitely many fixed points, in each case). Clearly $f \times g$ will then be a Lefschetz map of $M \times N$, with set of fixed points $Fix(f \times g) = Fix(f) \times Fix(g)$. Also, from the definition of $L_x f$ as $sign(det(df_x - I_{T_x M}))$ for $x \in Fix(F)$ (and similarly for g and $f \times g$), we see immediately that, for $(x, y) \in Fix(f) \times Fix(g)$:

$$L_{(x,y)}(f \times g) = L_x(f)L_y(g).$$

Thus:

$$\chi(M \times N) = \sum_{(x,y) \in Fix(f \times g)} L_{(x,y)}(f \times g) = (\sum_{x \in Fix(f)} L_x(f))(\sum_{y \in Fix(g)} L_y(g)) = \chi(M)\chi(N).$$

(ii) In general (with X, Z compact, oriented, with empty boundary), if $f : X \to Y, g : Z \to Y$ with $dim(X) + dim(Z) = dim(Y)$, $f \pitchfork g$ iff $f \times g \pitchfork \Delta_Y$, and $I(f, g) = (-1)^{dim(Z)} I(f \times g, \Delta_Y)$ (p. 114). Applying this to $f = g = i : Z \to Y$ (the inclusion map) when $Z \subset Y$ and $dim(Z) = (1/2)dimY$, we have:

$$I(Z, Z) = I(i, i) = (-1)^{dim(Z)} I(i \times i, \Delta_Y) = I(Z \times Z, \Delta_Y)$$

if $dim(Z)$ is even (since $i \times i$ is the inclusion map of $Z \times Z$). If $dim(Z)$ is odd, we know $I(Z, Z) = 0$ (intersection number in Y), while considering the general equality:

$$I(f \times g, \Delta_Y) = (-1)^{dim(X)dim(Z)} I(g \times f, \Delta_Y)$$

applied to the case $f = g = i$ (the inclusion of Z in Y), coupled with the fact $dim(Z)^2$ is also odd, we see that $I(Z \times Z, \Delta_Y) = 0$, so the claimed equality also holds in this case.