M562, SPRING 2023–Homework set 3–solutions

1. *(i)* $R^k \setminus M$ *is path-connected, if* $dim(M) \leq k-2$.

Given $p, q \in R^k \setminus M$, let $f: I \to R^k$ be a smooth path (for instance a line segment) from p to q. By the transversality extension theorem, there exists $g: I \to R^k$ transversal to M, coinciding with f in $\{0,1\}$, in particular $g(0) = p\, g(1) = q$. If $g(t) \in M$ for some $t \in I$, we have $dg(x)[T_t I] + T_{g(x)}M = R^k$; impossible, since the space on the left has dimension at most $1 + n \leq k-1$. Thus $g(t) \notin M$, for all $t \in I$, and g connects p to q in $R^k \setminus M$. □

(ii) $R^k \setminus M$ *is simply-connected, if* $dim(M) \leq k-3$.

It is enough to show that any $f: S^1 \to R^k \setminus M$ (smooth) is freely homotopic to a constant map. This is certainly true in R^k, so $\exists p_0 \in R^k \setminus M$ and a homotopy $H: S^1 \times I \to R^k$ with $H(x,0) = f(x), H(x,1) = p_0 \forall x \in S^1$. Now use the transversality extension theorem to find $G: S^1 \times I \to M$ coinciding with H on $S^1 \times \{0,1\}$, in particular also a homotopy from f to p_0. But if $G(x,t) \in M$ for some (x,t), transversality means: $dG(x,t)[T_{x,t)}(S^1 \times I)] + T_{G(x,t)}M = R^k$. And this is not possible, since the vector space on the left has dimension at most $2 + n \leq k-1$. Thus $G(x,t) \notin M \forall (x,t)$, and the homotopy G takes values in $R^k \setminus M$.

2. Given $p, q \in X$, let U_M contain both p and q. There is a path in U_M joining them, hence X is path connected. If $a: I \to X$ is a loop at $x_0 \in X$, $a(I)$ is compact, hence covered by finitely many of the open sets U_n, hence contained in a single U_N, and a is homotopic in U_N to the constant loop at x_0, hence homotopic to a constant in X. □

3. $Z = Y \cup X$, where X is the graph of $f(x) = \sin\frac{1}{x}$ over the interval $0 < x \leq \frac{1}{\pi}$, while Y is an embedded closed arc from the origin to the point $p = (\frac{1}{\pi}, 0) \in R^2$, intersecting X only at p. Both X and Y are path connected, as is their intersection $\{p\}$; hence Z is path-connected. But not locally connected, since an arbitrarily small open disk centered at the origin intersects X in a disjoint union of arcs of X; hence 0 has no local basis of connected open subsets of Z (in the induced topology from R^2.)

Now let $X_n = X \cap \{(x,y); x > \frac{1}{n\pi}\}, Z_n = Y \cup X_n, Z = \bigcup_{n \geq 1} Z_n$. This expresses Z as an increasing union of contractible subsets; unfortunately Z_n is not open in Z (the origin is the problem), so we can't use the result from problem 2 to conclude Z is simply-connected. Instead we proceed directly.

Let $c: I \to Z$ be a loop based at the origin. We claim $c(I)$ is contained in Z_N, for some finite N. (This is all we need, since Z_N is homeomorphic to $[0,1)$, hence contractible.) Otherwise, for arbitrarily large n, we may find $t_n < t_{n+1}$ in I so that $c(t_n) = (\frac{1}{n\pi}, 0), c(t_{n+1}) = (\frac{1}{(n+1)\pi}, 0)$. The image of c in $[t_n, t_{n+1}]$ contains the arc of the graph of f between these two points (since it is a path connected subset of Z containing these two points), hence there exists $s_n \in (t_n, t_{n+1})$ so that $c(s_n) = (\frac{2}{(2n+1)\pi}, 1)$. But then $\lim c(s_n) = (0,1)$, which is not a point of Z; contradiction, proving the claim.

4. Represent $X = S^1 \vee S^1$ (the 'figure-eight space') by two circles in R^2 labeled L and R, tangent at a common point x_0. Parameterize L (clockwise) and R (counterclockwise), by loops $f_a : I \to L, f_b : I \to R, f_a(0) = f_b(0) = x_0$. As for $\tilde{X}$, denote the interval $[n, n+1], n \in \mathbb{Z}$ on the x-axis by L_n, and the circle with radius 1/3 tangent to the x-axis at the point $(n, 0)$ by R_n. Parametrize each of the R_n counterclockwise, using the same map b from I used to parametrize R counterclockwise.

Define the map $p : \tilde{X} \to X$ by mapping the x-axis to L, via $p(x) = f_a(x-n)$ if $x \in [n, n+1]$. And map each point on R_n to the corresponding point on R (that is, with the same parameter value under the map f_b from I). This map p is clearly continuous, and the preimage of x_0 under p are the points $(n, 0), n \in \mathbb{Z}$, on the x-axis. It is easy to see p is a covering map: sufficiently small open intervals at points in L or R (other than x_0) have preimage a disjoint union of open intervals on the x-axis (resp. a disjoint union of open intervals, one in each R_n), with p restricted to each of those intervals defining a homeomorphism onto its image. And a preimage of a sufficiently small neighborhood of x_0 is also a disjoint union of open subsets of $\tilde{X}$, each mapped homeomorphically under p.

The fundamental group of X (with basepoint x_0) is $F_2\langle a, b\rangle$, the free group on two generators–the homotopy classes of the loops f_a and f_b, denoted by a and b (resp.) Considering loops in $\tilde{X}$ from the origin $\tilde{x}_0 \in \tilde{X}$, we see that the subgroup $H(\tilde{x}_0)$ is generated by elements of the form:

$$b^m, ab^m a^{-1}, a^2 b^m a^{-2}, a^{-1} b^m a, a^{-2} b^m a^2, \ldots \quad m \in \mathbb{Z}.$$

In other words, it is the normal subgroup of $F_2\langle a, b\rangle$ generated by the cyclic subgroup $\{b^m; m \in \mathbb{Z}\}$. Since $H = H(\tilde{x}_0)$ is a normal subgroup, the covering is regular. (The lifts of f_a are all open, the lifts of f_b all closed.)

The quotient group $F_2\langle a, b\rangle/H$ is the cyclic group $\{a^m, a \in \mathbb{Z}\}$ (any word in a, b can be reduced to one of those by repeated multiplication by elements of H); so this is the automorphism group of the cover, with a^m acting on $\tilde{X}$ via translation by the integer m (to the right, if $m > 0$.)

5. Let N be the common cardinality of each fiber of p. The map $f : \tilde{X} \to R$, being injective on fibers of p, defines a linear ordering in each fiber:

$$p^{-1}(x) = \{\tilde{x}_1 < \tilde{x}_2 < \ldots < \tilde{x}_N\}.$$

It is enough to show that $N = 1$. Let c be a loop at $x_0 \in X$, lift c to $\tilde{c}$ from $\tilde{x}_1^0$, the lowest point on the fiber at x_0. If $\tilde{c}$ is not closed, $\tilde{c}(1) = \tilde{x}_k^0$ with $2 \leq k \leq N$ (so necessarily $N \geq 2$). Let $p^{-1}(c(t)) = \{\tilde{x}_1(t) < \ldots < \tilde{x}_N(t)\}$. Consider the subsets of $I = [0, 1]$:

$$U = \{t \in I; \tilde{c}(t) < \tilde{x}_k(t)\}; \quad V = \{t \in I; \tilde{c}(t) > \tilde{x}_{k-1}(t)\}.$$

Note that: (i) both U and V are open in I (from the definition of covering map and continuity of $\tilde{c}$ and of f); (ii) $U \cap V = \emptyset$, for a t_0 in the intersection would satisfy $\tilde{x}_{k-1}(t_0) < \tilde{c}(t_0) < \tilde{x}_k(t_0)$, not possible; (iii) the union of U and V is I, for if $\tilde{c}(t) \leq \tilde{x}_{k-1}(t)$, then $\tilde{c}(t) < \tilde{x}_k(t)$. (iv) $0 \in U$ and $1 \in V$, so neither set is empty. This contradicts the fact $I = [0,1]$ is connected. So any lift to $\tilde{X}$ of a loop in X is closed, or $H(\tilde{x}_0^1) = \pi_1(X, x_0)$. But the index of $H(\tilde{x}_0^1)$ in $\pi_1(X, x_0)$ is exactly N (the cardinality of the fibers), so $N = 1$, and p is a homeomorphism.

Remark: Note the conclusion of the exercise is false if the fibers are infinite, as the standard exponential covering from R to S^1 shows..

6. Define $a : I \to S^1$ via $a(t) = f(e^{2\pi it})$, and denote by F a lift of a to R, over the standard exponential covering $R \to S^1$. The degree of f is the integer d so that:

$$F(t+1) = F(t) + d, \qquad \text{where } f(e^{2\pi it}) = e^{2\pi iF(t)}, t \in R.$$

Since f is odd, we have:

$$f(e^{2\pi i(t+1/2)}) = e^{\pi i} f(e^{2\pi it}), \text{ or } F(t+1/2) = F(t) + m/2 \quad \forall t \in R,$$

where $m \in \mathbb{Z}$ is an odd integer. But this implies $F(t+1) = F(t) + m$ $\forall t \in R$. Thus $d = m$, an odd integer.

7. The map g exists iff $f : U \to \mathbb{C} \setminus \{0\}$ lifts to $\mathbb{C}$ over the standard covering map $exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$. By the lifting criterion, this happens iff the image of $\pi_1(U)$ under f_* is trivial, or equivalently iff $f \circ c$ is homotopic to a constant in $\mathbb{C} \setminus \{0\}$, for any loop c in U. But the loop $f \circ c$ is homotopic to a constant loop in $\mathbb{C} \setminus \{0\}$ iff its degree with respect to $0 \in \mathbb{C}$ is zero.