REMARKS ON THE MOD 2 DEGREE.

Let $f : X \to Y$ be a smooth map, where $\dim X = \dim Y$; X is compact (with *empty* boundary) and Y is connected, without boundary. If $y \in Y$ is a regular value for f (equivalently, f is transversal to the 0-dim submanifold $\{y\}$ of Y), let $deg_2(f, y) = \#f^{-1}(y)$ mod 2. This makes sense, since in this case $f^{-1}(y)$ is a finite set (since f is a local diffeomorphism at each point of the preimage of y, and X is compact.) If $y \in Y$ is not a regular value, find $g : Y \to X$ smooth and homotopic to f, so that y is a regular value for g; then $g^{-1}(y)$ is finite, and its cardinality mod 2 is independent of the particular g chosen (by homotopy invariance of the mod 2 intersection number; or directly, since one-dimensional compact manifolds with boundary have an even number of boundary points.) So we set $deg_2(f, y) = \#g^{-1}(y)$ for any such g. The theorem on p. 80 of [G-P] states, in equivalent form:

Theorem. $deg_2(f, y)$ is independent (mod 2) of the point $y \in Y$ chosen.

This result is true, but the proof given on p.81 of [G-P] is slightly misleading: it may appear to be based on connectedness of Y and the fact that $\#f^{-1}(y)$ is (finite and) locally constant, which is true only for y in the set of regular values, a set which is not always connected, even if Y is. And it is not true that $\#f^{-1}(y)$ is constant over all $y \in Y$, even mod 2. Consider the example:

Example. $X = S^1 \subset \mathbb{R}^2, Y = \mathbb{R}, f : X \to Y, f(x_1, x_2) = x_1$. Any $y \neq \pm 1$ is a regular value. But $\#f^{-1}(\pm 1) = 1$, while $\#f^{-1}(y) = 2$ if $|y| < 1$ and $\#f^{-1}(y) = 0$ if $y < -1$ or $y > 1$ (so it's true that $\#f^{-1}(y) = 0$ mod 2, if y is a regular value.) Here we have $deg_2(f) = 0$.

In fact it is easy to find examples of $f : S^1 \to \mathbb{R}$ smooth, so that for some $y \in \mathbb{R}$, $f^{-1}(y)$ contains countably many nondegenerate intervals (so it is very much an infinite set.)

A better proof is given in J. Milnor's *Topology from the differentiable viewpoint*, p.20–25. It leads to a different (but equivalent) definition of the mod 2 degree. Here is an outline. The proof relies on the following lemma:

Homogeneity Lemma. Let y, z be arbitrary interior points of the connected manifold N. Then there exists a smooth diffeomorphism $h : N \to N$, isotopic to the identity (that is, homotopic to the identity via maps which are diffeomoprhisms), so that $h(y) = z$.

Theorem. Let X, Y be manifolds of the same dimension, with X compact, without boundary, and Y connected. Then if y, z are regular values of f, we have:

$$\#f^{-1}(y) \equiv \#f^{-1}(z) \text{ mod } 2.$$

This common residue class mod 2, called the mod 2 degree of f, $deg_2(f)$, depends only on the homotopy class of f.

Proof. Let h be a diffeomorphism of Y isotopic to the identity, with $h(y) = z$.

Then z is a regular value of $h \circ f$, which is homotopic to f. Thus:

$$\#(h \circ f)^{-1}(z) \equiv \#f^{-1}(z) \text{ mod } 2.$$

But since $(h \circ f)^{-1}(z) = f^{-1}(h^{-1}(z)) = f^{-1}(y)$, we have $\#(h \circ f)^{-1}(z) = \#f^{-1}(y)$, and therefore:

$$\#f^{-1}(y) \equiv \#f^{-1}(z) \text{ mod } 2.$$

Define $deg_2 f$ to be this common cardinality of the preimage of a regular value, mod 2.

If f is (smoothly) homotopic to g, let $y \in Y$ be a regular value for both f and g (using Sard's theorem). Then $\#f^{-1}(z) \equiv \#g^{-1}(z)$ (from the fact one-dimensional compact manifolds with boundary have an even number of boundary points). Hence $deg_2 f = deg_2 g$. □

Proof of the homogeneity lemma (outline).

(i) Let $B \subset R^n$ be the open unit ball, $\varphi : R^n \to [0, \infty)$ a smooth function, positive on B and zero on $R^n \setminus B$. For a unit vector $c \in S^{n-1}$, consider the smooth, bounded vector field on R^n: $X_c(x) = \varphi(x)c$ (which vanishes outside of B). Let $\{F_t\}_{t \in R}$ be the flow of X_c, a one-parameter group of smooth diffeomorphisms of R^n, all isotopic to the identity $F_0 = Id$, and equal to the identity outside of B. ($F_t(x)$ is the value at time t of the solution of the system of ODE defined by X_c, with initial condition $x \in R^n$.) Given $z_0 \in B$, we may find c and t so that $F_t(0) = z_0$.

(ii) Define an equivalence relation in the interior of N: $y \sim z$ if there exists a diffeomorphism isotopic to the identity taking y to z. Since each point $y \in N$ has a neighborhood diffeomorphic to $B \subset R^n$ (via a chart taking y to 0), part (i) shows the equivalence classes are open sets. Since N is connected, there is only one equivalence class. □