Algebraic Topology
The Grassmann Manifold as a CW-complex

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Adapted from <u>Characteristic Classes</u> by Milnor and Stasheff

We start with the definition of the Grassmann Manifold $G_n\left(\mathbb{R}^{n+k}\right)$ and its topology.

Definition. *The **Grassmann Manifold** $G_n\left(\mathbb{R}^{n+k}\right)$ is the set of all n-dimensional planes through the origin of $\mathbb{R}^{n+k}$ (or all n-dimensional subspaces of $\mathbb{R}^{n+k}$).*

Definition. *An **n-frame** in $\mathbb{R}^{n+k}$ is an n-tuple of linearly independent vectors of $\mathbb{R}^{n+k}$. The collection of all n-frames in $\mathbb{R}^{n+k}$ is an open subset of $\mathbb{R}^{n+k}\times\cdots\times\mathbb{R}^{n+k}$ (where the product is over n terms) called the **Steifel Manifold** $V_n\left(\mathbb{R}^{n+k}\right)$.*

We will view $G_n\left(\mathbb{R}^{n+k}\right)$ as a quotient space of $V_n\left(\mathbb{R}^{n+k}\right)$. Define the function

$$q : V_n\left(\mathbb{R}^{n+k}\right) \to G_n\left(\mathbb{R}^{n+k}\right)$$

by the rule

$$q(\{v_1, \dots, v_n\}) \mapsto \operatorname{span}\{v_1, \dots, v_n\}$$

and give $G_n\left(\mathbb{R}^{n+k}\right)$ the quotient topology, where $U \subset G_n\left(\mathbb{R}^{n+k}\right)$ is open if and only if $q^{-1}(U)$ is open in $V_n\left(\mathbb{R}^{n+k}\right)$.

Note that if $V_n^0\left(\mathbb{R}^{n+k}\right) \subset V_n\left(\mathbb{R}^{n+k}\right)$ is the set of all orthonormal n-frames, we can restrict the above quotient map to this subspace and view $G_n\left(\mathbb{R}^{n+k}\right)$ as a quotient space of $V_n^0\left(\mathbb{R}^{n+k}\right)$.

We note without proof that the Grassman Manifold is a compact topological manifold of dimension nk.

We now define the Schubert Symbol, which is a method of capturing ‘when a subspace gains its dimensions’.

Given $\mathbb{R}^m$, we identify lower dimensional Euclidean spaces in the the obvious way, that is we have

$$\mathbb{R}^0 \subset \mathbb{R}^1 \subset \cdots \subset \mathbb{R}^{m-1} \subset \mathbb{R}^m$$

where

$$\mathbb{R}^k = \{v \in \mathbb{R}^m : v = (v_1, \dots, v_k, 0, \dots, 0)\}.$$

Given any n-plane $X \subset \mathbb{R}^m$ we have a sequence of integers

$$0 \le \dim\left(X \cap \mathbb{R}^1\right) \le \dim\left(X \cap \mathbb{R}^2\right) \le \cdots \le \dim\left(X \cap \mathbb{R}^{m-1}\right) \le \dim\left(X \cap \mathbb{R}^m\right) = n.$$

Note that two consectuive integers in such a sequnce differs by at most 1. To see this, consider the sequence of maps

$$X \cap \mathbb{R}^{k-1} \hookrightarrow X \cap \mathbb{R}^k \xrightarrow{k-\text{coordinate}} \mathbb{R}.$$

This is an exact sequence of maps (in that the kernel of the second map is exactly the image of the first), and are also linear maps. Note that the image of the second map has dimension at most one while the kernel has dimension equal to $\dim\left(X \cap \mathbb{R}^{k-1}\right)$. The claim then follows by the rank-nullity theorem.

Definition. *A **Schubert symbol** $\sigma = (\sigma_1, \dots, \sigma_n)$ is a sequence of n integers satisfying*

$$1 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_{n-1} < \sigma_n \leq m.$$

We want to view a given Schubert symbol as a possible way for an n-plane to 'gain dimensions'. For instance, suppose we are interested in 3-planes in $\mathbb{R}^5$. If $\sigma = (2, 3, 5)$, this will correspond 3-planes such that

$$\begin{aligned}
\dim(X \cap \mathbb{R}^1) &= 0 \\
\dim(X \cap \mathbb{R}^2) &= 1 \\
\dim(X \cap \mathbb{R}^3) &= 2 \\
\dim(X \cap \mathbb{R}^4) &= 2 \\
\dim(X \cap \mathbb{R}^5) &= 3.
\end{aligned}$$

For each Schubert symbol σ, we will denote by $e(\sigma) \subset G_n(\mathbb{R}^m)$ the set of all n-planes X such that

$$\dim\left(X \cap \mathbb{R}^{\sigma_i}\right) = i, \text{ and } \dim\left(X \cap \mathbb{R}^{\sigma_i - 1}\right) = i - 1.$$

(It is worth taking the time to verify that this matches the above example, and does in fact capture the information we seek.)

Note that this construction forms a partition of $G_n(\mathbb{R}^m)$, as each n-plane $X \in G_n(\mathbb{R}^m)$ is contained in precisely one of the sets $e(\sigma)$. The different sets of this partition will be the cells of our CW-complex.

We will denote by $\mathbb{R}^k_+ \subset \mathbb{R}^m$ the 'open' half-space

$$\mathbb{R}^k_+ = \{(v_1, \dots, v_k, 0, \dots, 0) \in \mathbb{R}^m : v_k > 0\}.$$

Lemma. *Let $\sigma = (\sigma_1, \dots \sigma_n)$ be a Schubert symbol. Each n-plane $X \in e(\sigma)$has a unique orthonormal basis $\{x_1, \dots, x_n\}$ such that $x_i \in \mathbb{R}^{\sigma_i}_+$ for $i = 1, \dots, n$.*

Proof. This is by construction (and induction). Suppose $X \in e(\sigma)$. Then we have that $\dim(X \cap \mathbb{R}^{\sigma_1}) = 1$ and $\dim\left(X \cap \mathbb{R}^{\sigma_1 - 1}\right) = 0$. We select a basis vector x_1 for $X \cap \mathbb{R}^{\sigma_1}$, and note that if we require it to be of unit length there are two choices. One of them has a positive entry in the σ_1 component, and one has a negative entry. (Note that it cannot be 0 by the dimensionality statements above). We choose the positive one, which is in $\mathbb{R}^{\sigma_1}_+$.

Continuing on, we have that $\dim(X \cap \mathbb{R}^{\sigma_2}) = 2$ and $\dim\left(X \cap \mathbb{R}^{\sigma_2 - 1}\right) = 0$. We have x_1 as our first basis vector, and if we require x_2 to be unit length and orthogonal to x_1, we again have two choices. One of them has a positive entry in the σ_2 component, and one has a negative entry. Again, we choose the positive on, which is in $\mathbb{R}^{\sigma_2}_+$. Continuing in this manner,we find that there exists an orthonormal basis $\{x_1, \dots, x_n\}$ with $x_i \in \mathbb{R}^{\sigma_i}_+$ for $i = 1, \dots, n$. Since each vector in the basis is uniquely determined at each step, this is the unique orthonormal basis satsifying this condition. □

We will define the set $e'(\sigma)$ to be the set of all orthonormal n-frames $\{x_1, \dots, x_n\}$ such that $x_i \in \mathbb{R}^{\sigma_1}_+$ for $i = 1, \dots, n$. The set $\overline{e'(\sigma)}$ will denote the set of all orthonormal n-frames $\{x_1, \dots, x_n\}$ such that $x_i \in \overline{\mathbb{R}^{\sigma_i}_+} = \mathbb{R}^{\sigma_i}_{\geq 0}$ for $i = 1, \dots, n$.

Lemma. *Let $\sigma = (\sigma_1, \dots \sigma_n)$. The set $\overline{e'(\sigma)}$ is homeomorphic to the closed disc $D^{d(\sigma)}$ where $d(\sigma) = (\sigma_1 - 1) + (\sigma_2 - 2) + \cdots + (\sigma_n - n)$. Furthermore, the quotient map q, maps the interior $e'(\sigma)$ homeomorphically onto $q(\sigma)$.*

Proof. This is done using induction on the Schubert symbol. Suppose $n = 1$, that is, $\sigma = (\sigma_1)$. Note that the set $\overline{e'(\sigma)}$ is

$$\overline{e'(\sigma)} = \{(x_1, x_2, \dots, x_{\sigma_1}, 0, \dots, 0) : \sum x_i^2 = 1, x_{\sigma_1} \geq 0\}.$$

Thus, $\overline{e'(\sigma)}$ is a closed hemisphere of dimension $\sigma_1 - 1$, and is therefore homeomorphic to the closed disc $D^{\sigma_1 - 1}$.

To prove the inductive step, we define a linear transformation.

Give two unit vectors $u, v \in \mathbb{R}^m$, with $u \neq -v$, let $T(u, v)$ denote the rotation of $\mathbb{R}^m$ sending u to v, and keeping fixed all vectors orthogonal to both u and v. The following are easily (though not necessarily quickly) verified, and are given without proof:

1. $T(u, v)x$ is continuous over u, v, and x.

2. If $u, v \in \mathbb{R}^k$, then $T(u, v)x \equiv x \pmod{\mathbb{R}^k}$

3. $T(u, v) \in SO(\mathbb{R}^m)$.

Now, given an orthonormal n-frame $(x_1, \dots, x_n) \in \overline{e'(\sigma)}$ we consider transformation

$$T = T(e_{\sigma_n}, x_n) \circ T(e_{\sigma_{n-1}}, x_{n-1}) \circ \cdots \circ T(e_{\sigma_1}, x_1),$$

where e_{σ_i} is the σ_i standard basis vector (with a 1 in the σ_i component and 0 in all other components). Note that since e_{σ_j} is orthogonal to x_i for $j > i$, (and the sets $\{e_{\sigma_i}\}$ and $\{x_i\}$ are orthonormal), the result of this transformation is to take the standard basis vectors e_{σ_i} to the given vectors x_i for $i = 1, \dots, n$.

Suppose inductively that we have that $\overline{e'(\sigma_1, \dots, \sigma_n)}$ is homeomorphic to D^{d_n} where $d_n = (\sigma_1 - 1) + (\sigma_2 - 2) + \cdots + (\sigma_n - n)$. Given $\sigma_{n+1} > \sigma_n$, we define the set

$$D = \{u \in \mathbb{R}^{\sigma_{n+1}}_{\geq 0} : |u| = 1, e_{\sigma_i} \cdot u = 0 \text{ for } i = 1, \dots, n\}.$$

Note that D is a closed hemisphere of dimension $\sigma_{n+1} - (n + 1)$. (One way to see this is to note that without the condition that $e_{\sigma_i} \cdot u = 0$, D would be a closed hemisphere of dimension $\sigma_{n+1} - 1$. Then note that for each basis vector to which you require orthogonality, you reduce the space by one dimension.)

Define the function

$$f : \overline{e'(\sigma_1, \dots, \sigma_n)} \times D \to \overline{e'(\sigma_1, \dots, \sigma_n, \sigma_{n+1})}$$

by the rule

$$f((x_1, \dots, x_n), u) = (x_1, \dots, x_n, Tu)$$

where T is the transformation defined above, depending continuously on $x_1, \dots, x_n$.

We first note that $(x_1, \dots, x_n, Tu)$ is in fact in the set $\overline{e'(\sigma_1, \dots, \sigma_n, \sigma_{n_1})}$. To do this, we note that since T is the composition of transformations in $SO(\mathbb{R}^m)$, it is itself in $SO(\mathbb{R}^m)$. The important upshot of this is that we have $Tv \cdot Tv = v \cdot v$ for all vectors v. Thus we have by the definition of D, that

$$x_i \cdot Tu = Te_{\sigma_i} \cdot Tu = e_{\sigma_o} \cdot u = 0$$

and

$$Tu \cdot Tu = u \cdot u = 1.$$

We also have that $Tu \in \mathbb{R}^{\sigma_{n+1}}_{\geq 0}$ since $Tu \equiv u \pmod{\mathbb{R}^{\sigma_n}}$ (the transformation T does not change the σ_{n+1} component of u, which is nonnegative by definition of D).

We know that this map is continuous since T is continuous. It also has a continuous inverse where given an orthonormal set $x_1, \dots, x_n, x_{n+1}$ in $\overline{e'(\sigma_1, \dots, \sigma_n, \sigma_{n+1})}$ we have

$$f^{-1}((x_1, \dots, x_n, x_{n+1})) = ((x_1, \dots, x_n), T^{-1}x_{n+1})$$

where T^{-1} is defined as

$$T^{-1} = T(x_1, e_{\sigma_1}) \circ T(x_2, e_{\sigma_2}) \circ \cdots \circ T(x_n, e_{\sigma_n}).$$

Note that $T^{-1}x_{n+1}$ is in D since T^{-1} is also in $SO(\mathbb{R}^m)$ and maps orthonormal sets to orthonormal set and therfore T^{-1} maps the orthonormal set $(x_1, \dots, x_n, x_{n+1})$ to the orthonormal set $(e_{\sigma_1}, \dots, e_{\sigma_n}, T^{-1}x_{n+1})$.

Thus $\overline{e'(\sigma_1, \dots, \sigma_n)} \times D$ is homeomorphic to $\overline{e'(\sigma_1, \dots, \sigma_n, \sigma_{n+1})}$, and by the inductive hypthosis, we have that $\overline{e'(\sigma_1, \dots, \sigma_n, \sigma_{n+1})}$ is homeomorphic to the closed disc $D^{d_{n+1}}$ where $c_{n+1} = (\sigma_1 - 1) + \dots + (\sigma_{n+1} - (n+1)$.

It then remains only to note that the defined homeomorphism f maps $e'(\sigma_1, \dots, \sigma_n) \times \mathrm{int}D$ to $e'(\sigma_1, \dots, \sigma_n, \sigma_{n+1})$. To see this, note that if an n-frame is in the boundary of $e'(\sigma_1, \dots, \sigma_n, \sigma_{n+1})$, then we must have that $\dim(X \cap \mathbb{R}^{\sigma_i}) < i$ for some $i = 1, \dots, n+1$. If this is true for $i = 1, \dots, n$, then its preimage under f must be in the boundary of $e'(\sigma_1, \dots, \sigma_n)$. If this is true for $n+1$, then the $n+1$ of the final basis vector is 0, and its preimage is in the boundary of D.

Finally, we show that q maps $e'(\sigma)$ homeomorphically onto $e(\sigma)$.

We know that q is continuous, being a quotient map.By the previous Lemma, we know that restricting q to $e'(\sigma)$ gives us a bijective map. We now show that it is a closed map, and therefore a homeomorphism.

Let $A \subset e'(\sigma)$ be a closed map in the subspace topology. Then $A = \overline{A} \cap e'(\sigma)$, where $\overline{A}$ is the closure of A in $V_n^0(\mathbb{R}^m)$. Note also that $\overline{A} \subseteq \overline{e'(\sigma)}$ since $\overline{e'(\sigma)}$ is a closed set containing A.

Since $\overline{e'(\sigma)}$ is homeomorphic to a closed disc, it is compact, so $\overline{A} \subseteq \overline{e'(\sigma)}$ is also compact.

Thus $q(\overline{A})$ is compact, being the continuous image of a compact set, and as $G_n(\mathbb{R}^m)$ is Hausdorff (being a topological manifold), we have that $q(\overline{A})$ is closed in $G_n(\mathbb{R}^m)$. It then follows that $q(\overline{A}) \cap e(\sigma)$ is closed in the subspace topology in $e(\sigma)$.

We show that $q(\overline{A}) \cap e(\sigma) = q(A)$ by noting that if $(x_1, \dots, x_n) \in \overline{A} \setminus A$, then $(x_1, \dots, x_n) \in \overline{e'(\sigma)} \setminus e'(\sigma)$. Thus, one of the vectors x_i lies in the boundary of $\mathbb{R}_+^{\sigma_i}$, which is $\mathbb{R}^{\sigma_i - 1}$. This impiles that $\dim\left(X \cap \mathbb{R}^{\sigma_i - 1}\right) \geq i$, and so $X \notin e(\sigma)$, where X is the n-planed spanned by $(x_1, \dots, x_n)$. □

We are now prepared to prove that the sets $e(\sigma)$ are a CW-complex on $G_n(\mathbb{R}^m)$.

Theorem. *The $\binom{m}{n}$ sets $e(\sigma)$ form the cells of a CW-complex on the space $G_n(\mathbb{R}^m)$. Taking the direct limit as $m \to \infty$ you can obtain an infinite CW-complex over $G_n(\mathbb{R}^\infty)$.*

Proof. By the second Lemma above, we have that $G_n(\mathbb{R}^m)$ is a disjoint union of open cells $e(\sigma)$ with varying dimensions. We need only show that any point in the boundary of a cell $e(\sigma)$ belongs to a cell $e(\tau)$ of lower dimension.

We first note that $q(\overline{e'(\sigma)} = \overline{e(\sigma)}$. We already have that $q(\overline{e'(\sigma)}) \subset \overline{q(e'(\sigma))} = \overline{e(\sigma)}$ by continuity. Since $\overline{e'(\sigma)}$ is compact, $q(\overline{e'(\sigma)})$ is closed and we have $e(\sigma) = q(e'(\sigma)) \subset q(\overline{e'(\sigma)})$, so $\overline{e(\sigma)} \subseteq (\overline{e'(\sigma)})$.

This implies that any n-plane $x \in \overline{e(\sigma)} \setminus e(\sigma)$ has a basis $(x_1, \dots, x_n) \in \overline{e'(\sigma)} \setminus e'(\sigma)$. This basis is orthonormal by definition of $\overline{e'(\sigma)}$, and we have $x_i \in \mathbb{R}^{\sigma_i}$ for $i = 1, \dots, n$, since taking the closure of $e'(\sigma)$ won't 'add coordinates' to the basis vectors. Thus we have that $\dim(X \cap \mathbb{R}^{\sigma_i}) \geq i$ for $i = 1, \dots, n$, and so the Schubert symbol $\tau = (\tau_1, \dots, \tau_n)$ corresponding to X satisfies $\tau_i \leq \sigma_i$ for $i = 1, \dots, n$.

Since $(x_1, \dots, x_n) \notin e(\sigma)$, at least one of the vectors x_j must actually be in $\mathbb{R}^{\sigma_j - 1}$, and so we have $\tau_j < \sigma_j$. This strict inequality gives us that $d(\tau) < d(\sigma)$ where d is the dimension of $e(\sigma)$ define above.

The final claim about $G_n(\mathbb{R}^\infty)$ follows immediately from the process of taking direct limits and the definition of the CW topology. □