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Notes on Problem 16 [Hatcher ch. 0]

$\mathbb{R}^\infty$: sequences of real numbers which are eventually 0

$= \{ x : \mathbb{N} \to \mathbb{R} \ ; \ \exists N ; \ x(n) = 0 \ \forall n \geq N \}$ $\mathbb{N} = \{0,1,2,\dots\}$

(w/ weak topology: $C \subseteq \mathbb{R}^\infty$ closed $\iff$ $C \cap \mathbb{R}^N$ closed $\forall N$

here $\mathbb{R}^N$ embeds in $\mathbb{R}^\infty$ as $\mathbb{R}^N = \{ x \in \mathbb{R}^\infty \mid x(n) = 0 \ \forall n \geq N+1 \}$

$\pi_N : \mathbb{R}^\infty \to \mathbb{R}^N$ sets the $x(n)$ w/ $n \geq N+1$ to zero.

For $x \in \mathbb{R}^\infty$, let $N(x)$ be the least $N \geq 0$ s.t. $x(n) = 0$ for $n > N$.

$\|x\|_{N(x)}$ is the (eucl.) norm of $\pi_{N(x)}(x)$ in $\mathbb{R}^{N(x)}$

$S^\infty = \{ x \in \mathbb{R}^\infty ; \ \|x\|_{N(x)} = 1 \}$ (w/ the induced weak topology)

Theorem S^∞ is contractible

Pf Let $e : S^\infty \to S^\infty$ be the shift: "equator"

$(x_0, x_1, x_2, \dots) \xmapsto{e} (0, x_0, x_1, x_2, \dots) \in \mathcal{E}^\infty = \{ x \in S^\infty ; x(0) = 0 \}$

(e is cont., injective and closed: if $C \subseteq S^\infty$ is closed, $C \cap S^N$ is closed, hence compact for each N; hence $e(C) \cap S^{N+1} = e(C \cap S^N)$ is compact, hence closed $\forall N \geq 0$)

Thus e is an embedding $S^\infty \to \underset{\approx\, e(S^\infty)}{\mathcal{E}^\infty} \subseteq S^\infty$ and also homotopic to Id_{S^∞}:

Let $f_t : S^\infty \to S^\infty$,

$f_t(x_0, x_1, x_2, \dots) = \mathcal{N}\big((1-t)(x_0, x_1, \dots) + t(0, x_0, x_1, \dots) \big)$

where $\mathcal{N} : \mathbb{R}^\infty \setminus \{0\} \to S^\infty$ is the normalization map $\mathcal{N}(x) = \frac{1}{\|x\|_{N(x)}} x$

Then $f_0 = \mathrm{Id}_{S^\infty}$ and $f_1 = e$ (so $\mathcal{E}^\infty$ is also a deformation retract of S^∞).

We now define a homotopy $(g_t)_{t \in I} : \mathcal{E}^\infty \to S^\infty$ w/ $g_0 =$ inclusion, $g_1 : \mathcal{E}^\infty \to e_0 = (1, 0, 0, \dots) \in S^\infty$.

$g_t(x_0, x_1, \dots) = (t, \sqrt{1-t^2}\, x_0, \sqrt{1-t^2}\, x_1, \dots)$

Then $h_t = \begin{cases} f_{2t} & 0 \leq t \leq 1/2 \\ g_{2t-1} & 1/2 \leq t \leq 1 \end{cases}$ is a homotopy from Id_{S^∞} to the const. map $e_0 = (1, 0, 0, \dots) \in S^\infty$.

Rk. A similar argument shows $\mathbb{R}^\infty \setminus \{0\}$ is contractible (unlike $\mathbb{R}^N \setminus \{0\}$.)