Cellular approximation theorem

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November 15, 2021

Definition. *If X and Y are CW complexes, then $f : X \to Y$ is a* cellular map *if for every n-skeleton $X^n \subset X$, $f(X^n) \subset Y^n$.*

Theorem. *Every map $f : X \to Y$ of CW complexes X and Y is homotopic to a cellular map. If f is already cellular on a subcomplex $A \subset X$, then this homotopy may be taken to be constant on A.*

To prove the theorem, we will appeal to the following Lemma:

Lemma. *If Z is a CW complex obtained by attaching a cell e^k to subspace W, then for any map $f : I^n \to Z$ there exists a homotopy*

$$f_t : (I^n, f^{-1}(e^k)) \to (Z, e^k) \text{ rel } f^{-1}(W)$$

from $f_0 = f$ to map f_1 for which there is a polyhedron $K \subset I^n$ so that

(1) $f_1(K) \subset e^k$ and $f_1|_K$ is piecewise linear with respect to an identification of e^k with $\mathbb{R}^k$.

(2) $f_1^{-1}(U) \subset K$ for some nonempty open set $U \subset e^k$.

Proof of Lemma. Identify e^k with $\mathbb{R}^k$ and consider the closed balls B_1, B_2 of radius 1 and 2, centered at $0 \in \mathbb{R}^k$. Note that since B_2 is compact in $\mathbb{R}^k$, $f^{-1}(B_2) \subset I^n$ will be compact and thus $f|_{f^{-1}(B_2)}$ is uniformly continuous. There then exists $\epsilon > 0$ so that $|x - y| < \epsilon \implies |f(x) - f(y)| < \frac{1}{2}$ for all $x, y \in f^{-1}(B_2)$. Further, we wish to restrict ourselves to values

$$\epsilon < \frac{1}{2}\mathrm{dist}(f^{-1}(B_1), I^n \backslash f^{-1}(\mathrm{int}\ B_2)).$$

Next, subdivide I^n into cubes each contained in some open ball of radius less than ϵ. We now define two sets:

$$K_1 = \text{union of all cubes meeting } f^{-1}(B_1),$$

$$K_2 = \text{union of all cubes meeting } K_1.$$

We then have

$$f^{-1}(B_1) \subset K_1 \subset K_2 \subset f^{-1}(B_2)$$

by construction of the sets K_1, K_2 and by choice of ϵ. Now, we subdivide each cube in K_2 into simplices inductively: assuming the faces of a cube in K_2 are already subdivided into simplices, add the center point of this cube as a vertex to each simplex.

Next, define the map $g : K^2 \to e^k$ which is equal to f on vertices and extended linearly on each simplex. Additionally, define another map $\phi : K^2 \to [0, 1]$ so that $\phi = 1$ on vertices in K_1, $\phi = 0$ on vertices in $K_2 \backslash K_1$, and ϕ is extended linearly in each simplex. Note that for a simplex σ contained in K_1,

$$\phi(\sum_{i=0}^{n} t_i v_i) = \sum_{i=0}^{n} t_i \phi(v_i) = \sum_{i=0}^{n} t_k = 1$$

for each point $\sum_{i=0}^{n} t_i v_i \in \sigma$. Since K_1 is subdivided into simplices, $\phi(K_1) = 1$. However, if we now let σ be a simplex contained in $K_2 \backslash K_1$, then

$$\phi(\sum_{i=0}^{n} t_i v_i) = \sum_{i=0}^{n} t_i \cdot 0 = 0 \implies \phi(\sigma) = 0.$$

Finally, we define a homotopy

$$f_t : K_2 \to e^k, \quad f_t = (1 - \phi t)f + (\phi t)g$$

from $f_0 = f$ to some map f_1. We see that $f_1|_{K_1} \equiv g|_{K_1}$ and f_t is the constant homotopy on simplices contained in $K_2 \backslash K_1$. Because $K_1 \subset \text{int}(K_2)$, we may then extend f_t to be constant on $\overline{I^n \backslash K_2}$, giving a homotopy

$$f_t : I^n \to Z \text{ rel } f^{-1}(Z \backslash \text{int } e^k)$$

from $f_0 = f$ to f_1. Note that f_t maps $f^{-1}(e^k)$ into e^k by construction, so we have a homotopy

$$f_t : (I^n, f^{-1}(e^k)) \to (Z, e^k) \text{ rel } f^{-1}(W).$$

We now claim that $f_1(\overline{I^n \backslash K_1})$ is disjoint form a neighborhood U of 0 in e^k. This will involve two steps.

- Since $f^{-1}(B_1) \subset K_2$, $f(I^n \backslash K_2)$ must be disjoint from B_1.
- Let σ be a simplex in K_2 not entirely contained in K_1. Since $K_2 \subset f^{-1}(B_2)$, $f(\sigma)$ must be contained in some ball B_σ of radius $\frac{1}{2}$. Since g is linear on each simplex, $g(\sigma)$ is convex. Because the vertices of $g(\sigma)$ are contained in B_σ and B_σ is also convex, we must have $g(\sigma) \subset B_\sigma$. Since f_t is an affine combination of f and g on K_2, it follows that $f_t(\sigma) \subset B_\sigma$ as well and, in particular, $f_1(\sigma) \subset B_\sigma$. Because $f^{-1}(B_1) \subset K_1$ but $\sigma \not\subset K_1$, $f(\sigma)$ contains points outside B_1, so B_σ has points outside B_1. Since the radius of B_σ is $\frac{1}{2}$, this implies that $0 \notin B_\sigma$ and thus $0 \notin f_1(\sigma)$.

The two facts together imply that $f_1(\overline{I_n \backslash K_1})$ does not contain 0. Because f_1 must be continuous and $\overline{I_n \backslash K_1}$ is compact, $f_1(\overline{I^n \backslash K_1})$ must be disjoint from some open set U containing 0. The claim is then proven. Setting $K = K_1$, the lemma is then proven since the claim implies that $f_1^{-1}(U) \subset K_1 = K$. $\square$

Now, we prove the theorem.

Proof of Theorem. We will work by induction on the dimension n of the skeleton X^n. If $n = 0$, then $f(X^0)$ is a discrete set of points in Y. Note that by construction of CW complexes, each path-component of Y must contain some 0-cell in Y. Thus, each image $f(p)$ of a point $p \in X^0$ may be joined by a path in Y to some 0-cell $q \in Y^0$.

Now, assume that $f(X^{n-1}) \subset Y^{n-1}$ and consider an n-cell $e^n \subset X$. Then $\overline{e^n} \subset X$ is compact, so $f(\overline{e^n}) \subset Y$ is as well. By Proposition A.1 from the Appendix, then $f(e^n) \subset f(\overline{e^n})$ meets a finite number of cells in Y. Let e^k be the cell of largest dimension in Y that $f(e^n)$ meets. We may assume $k > n$, otherwise f is already cellular on e^n. We now wish to apply the Lemma, considering $Z = Y^k$, $W = Y^k \backslash (\text{int } e^k)$ and composing f with the characteristic map $\Phi : I^n \to X^{n-1} \cup e^n$ of e^n (we may choose I^n as our domain instead of D^n since the spaces are homeomorphic). We then obtain a homotopy

$$g_t : (I^n, f^{-1}(e^k)) \to (Y^k, e^k) \text{ rel } \partial I^n,$$

since

$$\Phi(\partial I^n) \subset X^{n-1} \subset f^{-1}(Y^{n-1}) \subset f^{-1}(Y^k \backslash e^k).$$

Note that we have g_1 piecewise linear on some polyhedron $K \subset I^n$, so $g_1(K)$ is convex and is thus contained in the union of finitely many hyperplanes of dimension $n < k$. Because $U \subset e^k$ is open and $g_1^{-1}(U) \subset K$, there must be points in U that g_1 misses. We then have an induced homotopy

$$f_t : X^{n-1} \cup e^n \to Y^k \text{ rel } X^{n-1}$$

so that f_1 misses points in U. We may let $p \in e^k$ be one of these points and compose a deformation retraction of $Y^k \backslash \{p\}$ onto $Y^k \backslash e^k$ with our homotopy f_t. Since $f(e^n)$ intersects finitely many cells, we may iterate this process a finite number of times, until $f_1(e^n)$ does meet meet any cells of dimension greater than n.

We now simultaneously apply this process on each n-cell $e^n \subset X$, except those contained in A. Note that these maps are compatible, so this gives a homotopy of $f|_{X^n}$ rel $X^{n-1} \cup A^n$ to a cellular map, where A^n is the n-skeleton of A. Using Proposition 0.16, we may then extend the homotopy to all of X, constant on $X \backslash X^n$. We may apply each homotopy $f_t : X^n \to Y$ on the interval $[1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}]$, since each X^n is stationary after the n-th interval. $\square$