TWO DEFINITIONS OF THE HOPF INVARIANT

Let $X = S^{2n-1}, Y = S^n$. The Hopf invariant of a map $f : X \to Y$, an integer depending only on the homotopy class of f, can be defined in two different ways, both using the cohomology cup product. The purpose of this note is to explain why they're equivalent.

1. The first definition appeals to the *attachment exact sequence* for f, in singular cohomology. Namely, denote by $Y_f = Y \sqcup_f D^{2n}$ the attachment space defined by attaching a $2n$-cell to S^n, with f as attachment map:

$$\cdots \to H^{p-1}(X) \xrightarrow{\Delta} H^p(Y_f) \xrightarrow{j^*} H^p(Y) \xrightarrow{f^*} H^p(X) \to \cdots$$

Here $j : Y \hookrightarrow Y_f$ is the inclusion map, so on the level of cochains $j^* : C^p(Y_f) \to C^p(Y)$ may be thought of as restriction of cochains (to chains taking values in the subspace Y of Y_f.)

For $p = n$, this exact sequence and the cohomology of X and Y implies $H^n(Y_f) \xrightarrow{j^*} H^n(Y) = \mathbb{Z}$ is *iso*; let $[\alpha] \in H^n(Y_f)$ be a generator, for a cocycle $\alpha \in C^n(Y_f)$.

For $p = 2n$, again from the cohomology of X, Y we find $\Delta : H^{2n-1}(X) \to H^{2n}(Y_f)$ is *iso*, so let $[\beta]$ be a generator, $\beta \in C^{2n}(Y_f)$. We have:

$$[\alpha \cup \alpha] = H^1_f[\beta] \quad \text{in } H^{2n}(Y_f),$$

for some $H^1_f \in \mathbb{Z}$. This is the first ('geometric') definition of the Hopf invariant H^1_f.

2. The second definition uses the action of f in the cohomology of X and Y directly. Let $a \in C^n(Y), b \in C^{2n-1}$ be cocycles whose classes $[a], [b]$ generate the cyclic abelian groups $H^n(Y), H^{2n-1}(X)$ resp. (We use $\mathbb{Z}$ coefficients throughout.)

The cohomology of X and Y implies the existence of cochains $c \in C^{2n-1}(Y), e \in C^{n-1}(X)$ so that:

$$\delta c = a \cup a \quad \text{and } \delta e = f^* a.$$

One checks easily that $z = f^*c - e \cup f^*a \in C^{2n-1}(X)$ is a cocycle. Then define $H^2_f \in \mathbb{Z}$ via:

$$[z] = H^2_f[b] \text{ in } H^{2n-1}(X).$$

3. Since $\Delta : H^{2n-1}(X) \to H^{2n}(Y_f)$ is *iso*, we may assume $\Delta[b] = [\beta]$. And we may also assume $j^*[\alpha] = [a]$; as cocycles, this means $a \in C^n(Y)$ is $\alpha \in C^n(Y_f)$ restricted to n-chains taking values in $Y \subset Y_f$.

Thus $H^1_f = H^2_f$ iff $\Delta[z] = [\alpha \cup \alpha]$ in $H^{2n}(Y_f)$. To show this we need to understand how the connection map Δ in the attachment exact sequence of f

is defined, on the level of cocycles. This requires consideration of the *mapping cylinder* of f:

$$M_f = Y \sqcup_f (X \times [0,1]).$$

(Here $(x,0)$ and $f(x) \in Y$ are identified in the quotient space M_f). We have $i : X \hookrightarrow M_f$ (inclusion, as $t=1$) and $k : Y \hookrightarrow M_f$ included as a deformation retract, via $r : M_f \to Y$ (so r^* and k^* are inverse maps in cohomology) :

$$r(x,t) = f(x), t \in [0,1]; \quad r(y) = y, y \in Y.$$

In general there is a natural map $p : M_f \to C_f$ from the mapping cylinder to the *mapping cone* of f, which maps $X \times 1$ to the basepoint $p_0 \in C_f$. In the present case where X is a sphere, C_f is just the attachment space Y_f, and we may think of p_0 as the center of the disk D with boundary X. We have:

Proposition: [Massey p. 245]. The map $p : (M_f, X) \to (Y_f, p_0)$ induces isomorphisms in cohomology:

$$p^* : H^q(Y_f) \to H^q(M_f, X), \quad q \geq 1;$$

where we implicitly used the isomorphism $H^q(Y_f, p_0) \sim H^q(Y_f), q \geq 1$. In cohomology in dimension n, we can combine this information into a commutative square of isomorphisms:

$$\begin{array}{ccc} H^n(Y_f) & \xrightarrow{j^*} & H^n(Y) \\ \big\downarrow p^* & & k^* \big\updownarrow r^* \\ H^n(M_f, X) & \longrightarrow & H^n(M_f) \end{array}$$

On the level of cocycles, the top row is restriction to $Y \subset Y_f$ the bottom row can be regarded as inclusion:

$$C^n(M_f, X) = \{w \in C^n(M_f); w(c) = 0 \text{ if } c \in C_n(M_f) \text{ and } im(c) \subset X\};$$

The fact it is an isomorphism (in cohomology) follows from considering the cohomology long exact sequence for the pair (M_f, X).

4. Now move the cocycles a and c from Y to M_f, using the retraction $r : M_f \to Y$:

$$r^*a = a_f \in C^n(M_f), \quad r^*c = c_f \in C^{2n-1}(M_f).$$

Claim: f^*a is the restriction of the cocycle a_f to $X \hookrightarrow M_f$, that is, for $i^* : C^n(M_f) \to C^n(X)$ the restriction, we have $i^*(a_f) = f^*(a)$. And likewise, $i^*(c_f) = f^*(c)$.

Recall that $i^*(a_f)$ is the restriction, $i^*(a_f) = a_{f|X}$. To verify the claim, note that $a_f(c) = a(r \circ c)$, for any $c \in C_n(M_f)$. So suppose $c \in C_n(M_f)$ maps to $X \subset M_f$, $im(c) \subset X$. The claim is:

$$a(r \circ c) = f^*a(c) = a(f \circ c)$$

for any n-chain $c \in C_n(M_f)$ with $im(c) \subset X$. This holds if $im(c) \subset X$ implies $r \circ c = f \circ c$. But this follows from the fact that $r_{|X} = f$ (since on M_f, $(x, 0) \sim f(x)$, and r fixes Y pointwise.)

5. Consider now the commutative triangle of isomorphisms in cohomology of dimension $2n$:

$$\begin{array}{ccc} & & H^{2n}(Y_f) \\ & \overset{\Delta}{\nearrow} & \big\downarrow p^* \\ H^{2n-1}(X) & \xrightarrow{\delta_X} & H^{2n}(M_f, X) \end{array}$$

(See [Massey], p. 246. The bottom row comes from the long exact sequence in cohomology for the pair (M_f, X).)

The connection homomorphism $\Delta : H^{2n-1}(X) \to H^{2n}(Y_f)$ of the attachment exact sequence of f is given by (see [Massey], p.246):

$$\Delta = (p^*)^{-1} \circ \delta_X,$$

where $\delta_X : H^{2n-1}(X) \to H^{2n}(M_f, X)$ is the connection homomorphism of the cohomology exact sequence for the pair (M_f, X).

On the level of cocycles, how does δ_X operate on a cocycle $w \in C^p(X)$? Answer (see [Hatcher], p. 200/201): extend w to $w' \in C^p(M_f)$, and then the cocycle you want is $\delta w' \in C^{p+1}(M_f, X)$. So first we need to find a suitable extension of $z \in C^{2n-1}(X)$ to M_f.

6. We have $i^*(a_f) = f^*a = \delta e \in C^n(X)$. Let e' be an extension of e from X to M_f (for example by zero), so $i^*(e') = e$. Then $a_f - \delta e'$ is a cocycle in $C^n(M_f)$, vanishing on $C_n(X)$; so we may regard it as a cocycle in the relative cochain group $C^n(M_f, X)$. Its cohomology class $[a_f - \delta e'] \in H^n(M_f, X)$ corresponds (via the commutative square in **3.**) to $[a] \in H^n(Y)$ and to $[\alpha] \in H^n(Y_f)$.

Note that:

$$i^*(c_f - e' \cup a_f) = i^*c_f - (i^*e') \cup (i^*a_f) = f^*c - e \cup f^*a = z.$$

Thus $c_f - e' \cup a_f \in C^{2n-1}(M_f)$ is an extension of $z \in C^{2n-1}(X)$ to M_f. This is the key step in the proof.

The codifferential of this extension of z is (since $\delta c_f = a_f \cup a_f$):

$$\delta(c_f - e' \cup a_f) = a_f \cup a_f - (\delta e') \cup a_f = (a_f - \delta e') \cup a_f \in C^{2n}(M_f, X).$$

Now consider that, in $C^{2n}(M_f, X)$:

$$(a_f - \delta e') \cup a_f \sim (a_f - \delta e') \cup (a_f - \delta e') \quad \text{(cohomologous)},$$

since the difference is a coboundary:

$$(a_f - \delta e') \cup \delta e' = \pm\delta((a_f - \delta e') \cup e').$$

We conclude:

$$\delta_X[z] = [(a_f - \delta e') \cup a_f] = [(a_f - \delta e') \cup (a_f - \delta e')] \in H^{2n}(M_f, X).$$

This cohomology class in $H^{2n}(M_f, X)$ is the image under p^* of $[\alpha \cup \alpha] \in H^{2n}(Y_f)$. Thus:

$$\Delta[z] = (p^*)^{-1}([(a_f - \delta e') \cup (a_f - \delta e')] = [\alpha \cup \alpha] \in H^{2n}(Y_f),$$

as we wished to show.

Sources: This argument is based on [Prasolov], p.219, where it is given without much detail. Most of the background needed in the proof appears in the discussion of the mapping cylinder and cohomology exact sequence of an attachment map given in [Massey], p. 244-247. The equivalence of definitions is proposed as an exercise in [Vick], p. 136 and in [Greenberg-Harper], p. 207.