M663 9/12/2023 (Zoom) 1

(X, A)

<u>goal</u>: relative homology vs. homology of X/A

<u>Prop.</u> $x_0 \in X$ $\quad H_n(X, x_0) \approx \tilde{H}_n(X)$ $\begin{cases} H_n(X), & n>0 \\ 0, & n=0 \text{ (}X\text{ path-conn)} \end{cases}$

<u>Split</u> <u>exact</u> <u>sequence</u>

$$0 \to A \underset{p}{\overset{i}{\rightleftarrows}} B \underset{s}{\overset{j}{\rightleftarrows}} C \to 0 \qquad \text{short exact}$$

<u>TFAE</u>

[Hatcher, p. 147]

(a) $\exists\ p: B \to A \quad p \circ i = \mathrm{id}_A$

(b) $\exists\ s: C \to B \quad j \circ s = \mathrm{id}_C$

(c) $\exists\ B \to A \oplus C$ <u>iso</u> (w/ commuting diagram).

$$\begin{tikzcd} & B \arrow[dr, "j"] \arrow[d] & \\ A \arrow[ur, "i"] \arrow[r] & A \oplus C \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

(a) $\Rightarrow$ (c)

$b \mapsto (p(b), j(b))$

(b) $\Rightarrow$ (c)

$(a, c) \to i(a) + s(c)$

$A \oplus C \to B$

<u>Also</u> in this case.

$B \approx i(A) \oplus \ker p$.

$(b = i(p(b)) + b - i(p(b)))$

$j: \ker p \to C$ is <u>iso</u>

⌜pf. onto, and ass. $j(b) = 0$ and $p(b) = 0$ then $b = i(a)$, $p(i(a)) = 0$, $a = 0$, $b = 0$ ⌟

<u>proof of</u> prop [Vick, p. 48]

long exact seq. for (X, x_0) breaks up into short ones

$$0 \to H_n(x_0) \xrightarrow{i_\#} H_n(X) \to H_n(X, x_0) \xrightarrow{\partial} 0 \in H_{n-1}(x_0)$$

(monomorphism, since $\{x_0\}$ retract of X)

∂ is the zero map, since $\mathrm{im}\,\partial = \ker i_\# = 0 \in H_{n-1}(x_0)$

The short exact seq.

$$0 \to H_{n-1}(x_0) \underset{\alpha_\#}{\overset{i_\#}{\rightleftarrows}} H_n(X) \longrightarrow H_n(X, x_0) \longrightarrow 0$$

splits via $\alpha_\# : H_n(X) \longrightarrow H_n(x_0)$ $\quad \alpha : X \to \{x_0\}$ const. map.

$$\alpha_\# \circ i_\# = (\alpha \circ i)_\# = (\mathrm{id}_{\{x_0\}})_\# = \mathrm{id}_{H_n(x_0)}$$

so

$$H_n(X) = \underbrace{\mathrm{im}\, i_\#}_{\approx H_n(x_0)} \oplus \underbrace{\ker \alpha_\#}_{\approx H_n(X, x_0)}$$

$$\begin{tikzcd} & H_n(X) \arrow[dr, "i_\#"] \arrow[d, "\cong"] & \\ 0 \to H_n(x_0) \arrow[ur] \arrow[r] & H_n(x_0) \oplus H_n(X,x_0) \arrow[r] & H_n(X, \cdot) \end{tikzcd}$$

since ~~exact~~ $j_\# : \ker \alpha_\# \longrightarrow H_n(X, x_0)$ is iso

$\subset H_n(X)$

The reduced homology $\tilde{H}_n(X)$ equals $\ker \alpha_\# \subset H_n(X)$

since
- for $n > 0$ $\quad H_n(x_0) = 0$, so $\ker \alpha_\# = H_n(X)$
- for $n = 0$ $\quad H_0(x_0) \cong \mathbb{Z}$ and $\alpha_\#$ is the "augmentation homomorphism"

w/ kernel $0 \in H_0(X)$ $\qquad \sum_{x \in X} n_x x \longmapsto \sum n_x \in \mathbb{Z}$

$\qquad \in S_0(X)$

(X path conn.) so in this case

$$\ker \alpha_\# = 0 = \tilde{H}_0(X)$$

Thus

$$H_n(X, x_0) \approx \tilde{H}_n(X) \qquad \square$$

more generally

recall if $A \subset X$ is a retract (via $g : X \to A$)

$g_\# : H_n(X) \to H_n(A)$

Then $\quad H_n(X) = H_n(A) \oplus \ker g_\#$

via $\quad 0 \to H_n(A) \underset{g_\#}{\overset{i_\#}{\rightleftarrows}} H_n(X) \longrightarrow H_n(X, A) \to 0$

($g_\#$ splits this exact seq.)

so $\quad H_n(X) \approx H_n(A) \oplus H_n(X, A)$ if $A \subset X$ retract $\square$

Prop. If $A \subset X$ is a def: retract of X,
then $H_*(X, A) = 0$

Proof long exact seq.
[Vick, p.45]

$$\cdots \to H_n(A) \xrightarrow{i_\#} H_n(X) \xrightarrow{\pi_\#} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

$i_\#$ is iso (from htpy invariance)

i.e. epi ($\Leftrightarrow$ ker $\pi_\# = H_n(X)$, so $\pi_\#$ is the zero map)

mono ($\Leftrightarrow$ im $\partial = 0$, or ∂ is the 0 map, or ker $\partial = H_n(X,A)$)

thus $H_n(X,A) = \ker \partial = \operatorname{im} \pi_\# = 0$
(exactness).

Q) More generally, consider $A \subset X$ and the map of pairs.

$$(X, A) \xrightarrow{\pi} (X/A, a_0) \qquad a_0 = \pi(A)$$

when does this induce an iso in homology?

Relative homotopy invariance [Vick, p.46]

$f, g : (X, A) \to (Y, B)$ htopic as maps of pairs
$f_t : A \to B$, $t \in [0,1]$

Then $f_\# = g_\# : H_n(X, A) \to H_n(Y, B)$

Follows from: $P : S_n(X) \to S_{n+1}(Y)$
s.t. maps $S_n(A) \to S_{n+1}(B)$

Thus induces $P : S_n(X, A) \to S_{n+1}(Y, B)$

chain htpy $P\partial + \partial P = g_\# - f_\#$

Theorem [Vick, p. 50] $A \subset X$ closed (X: Hausdorff + normal, say cpt. Hausdorff)

Suppose A is a strong def retract of a closed nbd U

"(X, A) good pair" ($A \subset \text{int}\, U \subset U \underset{\text{closed}}{\subset} X$)

Then: $\pi: (X, A) \longrightarrow (X/A, a_0)$

induces ISO $H_n(X, A) \xrightarrow{\sim} H_n(X/A, a_0)$

(thus for good pairs $H_n(X, A) \approx \tilde{H}_n(X/A)$)

Pf uses:
- exact seq. for triples $B \subset A \subset X$
- excision thm
- lemma

Lemma (X, A) "good pair", X cpt. Hausdorff

(A: str. def retract of X).

Then $\{a_0\}$ is a str. def retract of X/A

Pf see [Vick] p. 49

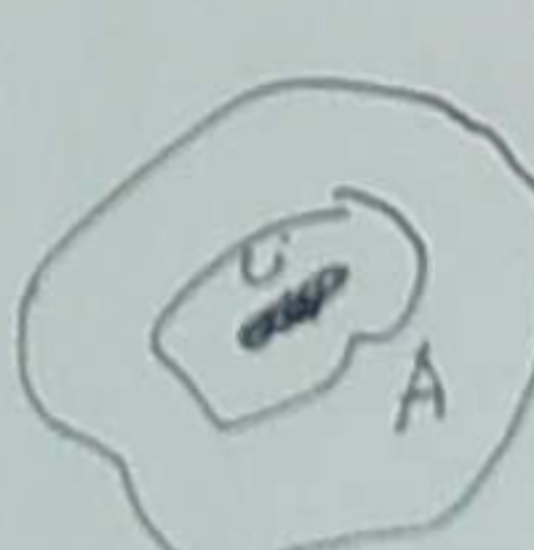
Excision thm [Vick, p. 47]

(X, A) pair, $U \subset A$ w/ $\overline{U} \subset A$

Then $i: (X \setminus U, A \setminus U) \longrightarrow (X, A)$

induces ISO $i_{\#}: H_n(X \setminus U, A \setminus U) \longrightarrow H_n(X, A)$

"excising U from (X, A) does not change rel. homology"



X

"H_* is a theory of boundaries"

Proof: next lecture

Long exact seq. for triples $(B \subset A \subset X)$
[Vick, p. 45]

$(B, A) \hookrightarrow (A, X)$ inclusion.

short exact seq

$$0 \to S_n(A, B) \longrightarrow S_n(X, B) \longrightarrow S_n(X, A) \longrightarrow 0$$

exact since : $S_n(X, A) = S_n(X)/S_n(A)$

group theory $\longrightarrow \cong \dfrac{S_n(X)/S_n(B)}{S_n(A)/S_n(B)} = \dfrac{S_n(X,B)}{S_n(A,B)}$

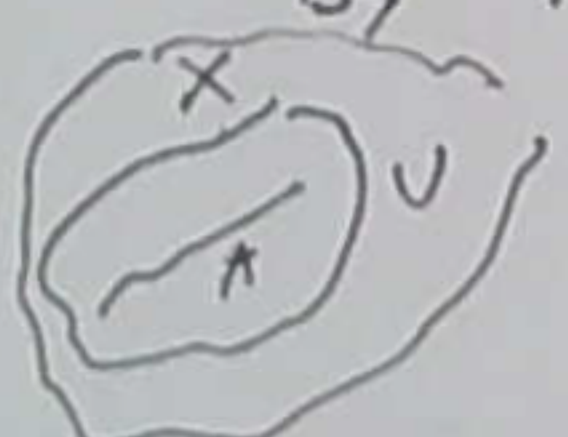
long exact seq of a triple $B \subset A \subset X$

$$\cdots \longrightarrow H_n(A, B) \longrightarrow H_n(X, B) \longrightarrow H_n(X, A) \xrightarrow{\partial} H_{n-1}(A, B) \longrightarrow \cdots$$

(X, A) good pair

$\boxed{\text{Proof of thm}}$ [Vick p. 50] $(X, A) \xrightarrow{\pi} (X/A, a_0)$ induces ISO in H_*

By the Lemma $\{a_0\}$ is a str. def. retract of $\pi(U)$ ⊛



exact seq. of triple $\{a_0\} \subset \pi(U) \subset X/A$.

$$\cdots \to H_n(\pi(U), a_0) \longrightarrow H_n(X/A, a_0) \longrightarrow H_n(X/A, \pi(U)) \to H_{n-1}(\pi(U), a_0)$$

$H_n(\pi(U), a_0) = 0$ (from ⊛), $H_{n-1}(\pi(U), a_0) = 0$ "

Thus $H_n(X/A, a_0) \xrightarrow{\simeq} H_n(X/A, \pi(U))$ is ISO (1)

Let V be open w/ $A \subset V$ w/ $\overline{V} \subset \mathrm{int}(U)$ (X is normal)

now excise V from the pair (X, U).

$H_n(X \setminus V, U \setminus V) \cong H_n(X, U)$ (2) (excision)

Consider the exact seq. for the triple $A \subset U \subset X$.

$$\cdots \to \underset{\substack{\| \\ 0}}{H_n(U,A)} \to H_n(X,A) \to H_n(X,U) \to \underset{\substack{\wr\| \\ 0}}{H_{n-1}(U,A)} \to \cdots$$

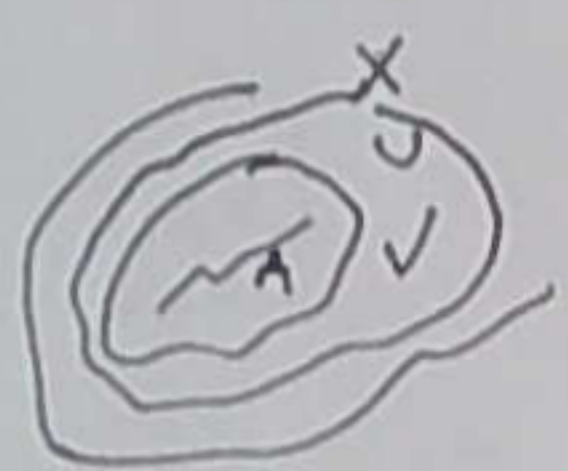
(A is a def. retract of U)

Thus $\quad H_n(X,A) \approx H_n(X,U)$. ③

Thus: $\quad H_n(X,A) \approx H_n(X\setminus V, U\setminus V)$. ④

Excise $\quad \pi(V)$ from $(X/A, \pi(U)\setminus\pi(V))$ $\quad$ (since $\overline{\pi(V)} \subset \pi(U)$)]

$$H_n(X/A, a_0) \overset{①}{\approx} H_n(X/A, \pi(U)) \approx H_n(X/A\setminus\pi(V), \pi(U)\setminus\pi(V))$$ ⑤



V is a nbd of the collapsed set A, so

$$\pi : (X\setminus V, U\setminus V) \to (X/A - \pi(V), \pi(U)\setminus\pi(V))$$

is a homeo of pairs.

[meaning: $\pi: X\setminus V \to X/A - \pi(V)$ is homeo
and restricts to $\pi: U\setminus V \to \pi(U)\setminus\pi(V)$. homeo]

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(U\setminus V) \arrow[r] \arrow[d, "\pi_\#"', "\text{iso}"] & H_n(X\setminus V) \arrow[r] \arrow[d, "\pi_\#"', "\text{iso}"] & H_n(X\setminus V, U\setminus V) \arrow[r] \arrow[d, dashed] & H_{n-1}(U\setminus V) \arrow[r] \arrow[d, "\pi_\#", "\text{iso}"'] & \cdots \\
\cdots \arrow[r] & H_n(\pi(U)\setminus\pi(V)) \arrow[r] & H_n(X/A\setminus\pi(V)) \arrow[r] & H_n(X/A - \pi(V), \pi(U)\setminus\pi(V)) \arrow[r] & H_{n-1}(\pi(U)\setminus\pi(V)) &
\end{tikzcd}$$

It follows from the "five lemma" — next page that

$$\pi_\# : H_n(X-V, U\setminus V) \to H_n(X/A - \pi(V), \pi(U)\setminus\pi(V)) \text{ is } \boxed{\text{iso}}$$

$$\underset{H_n(X,A)}{\wr\wr\ ④} \qquad\qquad \underset{H_n(X/A, a_0)}{\wr\wr\ ⑤}$$

Conclusion $\quad H_n(X,A) \approx H_n(X/A, a_0)$. $\quad \square$

FIVE LEMMA

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\varepsilon} \\ A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{l'} & E' \end{array}$$

Assume

- Rows are exact sequences.
- Diagram commutes.

Then $\alpha, \beta, \delta, \varepsilon$ ~~are~~ iso $\Longrightarrow$ γ iso

Pf [Hatcher] p. 129