9/14/23. ①

Thm X cpt. Hausdorff, $A \subset X$ closed

(X, A) 'good pair': $\exists\ U$ closed nbd of A in X s.t. U def. retracts to A

Then $\pi : (X, A) \longrightarrow (X/A,\ a_0)$ induces iso in H_*

(Cor: $H_n(X, A) \approx \tilde{H}_n(X/A)$)

Pf. Lemma $\{a_0\}$ is a def. retract of $\pi(U)$ in X/A

~~From~~ Thus: $H_n(\pi(U), a_0) = 0$

From the triple $\{a_0\} \subset \pi(U) \subset X/A$

$H_n(X/A, a_0) \xrightarrow{\sim} H_n(X/A, \pi(U))$ is iso ①

Analogously, in $A \subset U \subset X$, $H_n(U, A) = 0$, so

$H_n(X, A) \xrightarrow{\sim} H_n(X, U)$ is iso ②

now let $V \subset X$ open nbd of A w/ $\overline{V} \subset \text{int}\, U$

and excise V from the pair (X, U)

$H_n(X \setminus V,\ U \setminus V) \approx H_n(X, U)$ ③

In $X \setminus A$, excise $\pi(V)$ from $(X/A, \pi(U))$.

Then $H_n(X/A \setminus \pi(V),\ \pi(U) \setminus \pi(V)) \approx H_n(X/A, \pi(U))$ ④

Main observation Since V is a nbd of the collapsed set A,

$\pi : (X \setminus V,\ U \setminus V) \longrightarrow (X/A - \pi(V),\ \pi(U) \setminus \pi(V))$

is a homeo of pairs.

π induces iso in rel. homology:

Thus by the five lemma

$$\pi_* : H_n(X \setminus V,\ U \setminus V) \xrightarrow{\sim} H_n(X/A - \pi(V),\ \pi(U) \setminus \pi(V))$$ ⑤

$$\begin{array}{ccc} H_n(X \setminus V,\ U \setminus V) & & H_n(X/A - \pi(V),\ \pi(U) \setminus \pi(V)) \\ \wr\wr\ (3) & & \wr\wr\ (4) \\ H_n(X, U) & & H_n(X/A, \pi(U)) \\ \wr\wr\ (2) & & \wr\wr\ (1) \\ H_n(X, A) & & H_n(X/A, a_0) \end{array}$$

Corollary 'relative homeo' $f : (X, A) \longrightarrow (Y, B)$ ~~(cpt. Hausdorff pairs)~~

s.t. $f\ X \setminus A \xrightarrow{f} Y \setminus B$ is bijective ~~both~~

Then if X, Y cpt. Hausdorff, $(X, A), (Y, B)$ "good pairs": $f_* : H_*(X, A) \to H_*(Y, B)$ iso

Pf

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & Y \arrow[d, "\pi'"] \\ X/A \arrow[r, "f'"] & Y/B \end{tikzcd}$$

Let $f' = \pi' \circ f \circ \pi^{-1}$ (well-def, cts.)

f' is bijective from cpt. Hausdorff sp., hence homeo

$$\begin{tikzcd} H_n(X, A) \arrow[r, "f_*"] \arrow[d, "\pi_*"', "\text{iso}"] & H_n(Y, B) \arrow[d, "\pi'_*", "\text{iso (from thm)}"'] \\ H_n(X/A, a_0) \arrow[r, "f'_*"] & H_n(Y/B, b_0) \end{tikzcd}$$

iso (since f' homeo)

Hence f_* is iso

Examples

1) $\exists$ a rel. homeomorphism

$$f : (D^n, S^{n-1}) \longrightarrow (S^n, z) \qquad (z : \text{any pt.})$$

sat. the hypotheses. Thus

$$f_* : H_*(D^n, S^{n-1}) \longrightarrow H_*(S^n, z) \cong \widetilde{H}_*(S^n)$$

is <u>iso</u>

2) (Attaching an n-cell)

<u>Lemma</u> $f : S^{n-1} \longrightarrow Y$ (Y cpt. Hausdff)

Y_f : attachment space (of a D^n to Y, via f)

Then $Y \hookrightarrow Y_f$ is a str.-def retract of a cpt. nbd of Y in Y_f.

Thus

$$\begin{tikzcd} D^n \arrow[r] \arrow[rr, bend right, "h"'] & D^n \sqcup Y \arrow[r, "\pi"] & Y_f \end{tikzcd}$$

now gives $h : (D^n, S^{n-1}) \longrightarrow (Y_f, Y)$

inducing an <u>iso</u> $h_* : H_n(D^n, S^{n-1}) \longrightarrow H_n(Y_f, Y)$

(so $H_*(Y_f, Y)$ is free abelian (cyclic), w/ one basis element in dimension n).

① Excision theorem

(X, A) pair, $U \subset A$ w/ $\overline{U} \subset \text{int } A$.

Then the inclusion $i: (X \setminus U, A \setminus U) \longrightarrow (X, A)$

induces an iso in rel. homology $i_*: H_*(X \setminus U, A \setminus U) \longrightarrow H_*(X, A)$

"U can be excised from the pair (X, A) w/ no change in homology"

(since homology is a theory of boundaries).

Proof $\mathcal{U} = \{X \setminus U, \text{Int } A\}$ whose interiors cover X.

$\mathcal{U}' = \{A \setminus U, \text{Int } A\}$ whose interiors cover A

$\therefore$ $i: S_*^{\mathcal{U}}(X) \to S_*(X)$, $i': S_*^{\mathcal{U}'}(A) \to S_*(A)$

induce iso on homology

Now $S_*^{\mathcal{U}'}(A) \subset S_*^{\mathcal{U}}(X)$ (subcomplex), so $\exists$ a chain mapping of complexes

$$j: S_*^{\mathcal{U}}(X)/S_*^{\mathcal{U}'}(A) \longrightarrow S_*(X)/S_*(A) = S_*(X, A),$$

and by the "five lemma" j induces an iso on homology:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_n(S_*^{\mathcal{U}'}(A)) & \longrightarrow & H_n(S_*^{\mathcal{U}}(X)) & \longrightarrow & H_n(S_*^{\mathcal{U}}(X)/S_*^{\mathcal{U}'}(A)) & \longrightarrow & H_{n-1}(S_*^{\mathcal{U}'}(A)) & \to \cdots \\
& \wr\downarrow i'_\# & & \wr\downarrow i & & j_\# \downarrow & & \downarrow i'_\# & \\
\cdots \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A) & \longrightarrow & H_{n-1}(A) & \to \cdots
\end{array}$$

Note we have

$$S_*^{\mathcal{U}}(X) = S_*(X \setminus U) + S_*(\text{Int } A) \quad \text{(not nec. direct)}$$

$$S_*^{\mathcal{U}'}(A) = S_*(A \setminus U) + S_*(\text{Int } A) \quad \text{"}$$

Thus (general gp. theory)

$$S_*^{\mathcal{U}}(X)/S_*^{\mathcal{U}'}(A) = S_*(X \setminus U)/S_*(A \setminus U) = S_*(X - U, A - U),$$

with homology $H_n(X \setminus U, A \setminus U)$. So the iso $j_\#$ is

$$j_\#: H_n(X \setminus U, A \setminus U) \xrightarrow{\sim} H_n(X, A) \quad \text{(induced by inclusion).}$$