Singular homology

$$\begin{pmatrix} \text{top. spaces } X \\ \text{cont. maps} \\ f: X \to Y \end{pmatrix} \underset{f_\#}{\overset{H_*(X)}{\rightsquigarrow}} \begin{pmatrix} \text{abelian groups} \\ \text{homomorphisms} \end{pmatrix}$$

sing. simplex $\quad \sigma : \Delta_n \longrightarrow X$ cont.

$S_n(X)$: chain group : freely gen'd by n-simplices

$\partial_n : S_n(X) \longrightarrow S_{n-1}(X)$

$\partial_{n-1} \circ \partial_n = 0 \qquad \dfrac{\ker \partial_n}{\operatorname{im} \partial_{n+1}} = \dfrac{Z_n(X)}{B_n(X)} \overset{\text{def}}{=} H_n(X)$

$f : X \longrightarrow Y$.

$f_\# : S_n(X) \longrightarrow S_n(Y)$

$\sigma : \cancel{\Delta_n\ X} \ \Delta_n \longrightarrow X \qquad f_\# \sigma = f \circ \sigma : \Delta_n \longrightarrow Y$.

chain map $\qquad (S_n(X), \partial)_{n \geq 0}$: chain complex

$f_\# \circ \partial = \partial \circ f_\#$

hence induces hom: $\quad f_\# : H_n(X) \longrightarrow H_n(Y)$

$X \xrightarrow{f} Y \xrightarrow{g} Z$

$(g \circ f)_\# = g_\# \circ f_\# : H_n(X) \longrightarrow H_n(Z)$

consequence

f homeomorphism $\Longrightarrow$ $f_\#$ isomorphism.

homeomorphism invariance

Properties

- $H_n(\text{point}) = 0, \; n > 0$
  $H_0(\text{point}) \approx \mathbb{Z}$
- $H_0(\overset{\text{path}}{\text{connected}}) \approx \mathbb{Z}$ $\qquad X \neq \emptyset$

recall

$$\cdots \xrightarrow{\partial} S_1(X) \xrightarrow{\partial} S_0(X) \xrightarrow{\partial} 0$$

$$S_1(X) \xrightarrow{\partial} S_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0 \text{ (onto)}$$

$$\varepsilon\Big(\sum_{n_x \in \mathbb{Z}} n_x x\Big) \longrightarrow \sum n_x \in \mathbb{Z}$$

$\left\{ \begin{array}{l} \varepsilon \text{ is } \underline{\text{onto}} \\ \text{and } \ker \varepsilon = B_0(X) \end{array} \right.$ $\quad$ (so $\varepsilon \circ \partial_1 = 0$ on $S_1(X)$)

$$H_0(X) = \frac{S_0(X)}{B_0(X)} \approx \mathbb{Z} \quad \text{(1st iso of gr thy)}$$

reduced homology

consider the chain complex $(S_*(X), \tilde{\partial})$ $\qquad$ $\tilde{\partial} = \partial$ on $S_n(X)$, $n > 0$; $\tilde{\partial} = \varepsilon$ on $S_0(X)$

its homology $\qquad \tilde{H}_n(X) = \dfrac{\ker \tilde{\partial}_n}{\operatorname{im}(\tilde{\partial}_{n+1})}$

sat.

$$\left.\begin{array}{l} \tilde{H}_n(X) = H_n(X) \quad n > 0 \\ \tilde{H}_0(X) = \dfrac{\ker \varepsilon}{\operatorname{im} \partial_1} = 0 \end{array}\right\} \text{"reduced homology grs"}$$

$\longrightarrow$ X path-connected

Homotopy invariance

Thm $f, g : X \to Y$

If $f \simeq g$ then $f_\# = g_\#$ on : $H_n(X) \to H_n(Y)$

Proof From $F : X \times I \to Y$ $f_0 = f$, $f_1 = g$

construct

$$P : S_n(X) \to S_{n+1}(Y)$$

w/

$$P\partial + \partial P : S_n(X) \to S_n(Y).$$

s.t. $P\partial + \partial P = g_\# - f_\#$ (on S_*) "chain homotopy"

Then if $c \in Z_n(X)$ $\partial c = 0$

$$g_\# c - f_\# c = P(\partial c) + \partial(Pc) = \partial(Pc)$$

so $g_\# c - f_\# c \in B_n(Y)$, so $g_\#[c] = f_\#[c]$ in $H_n(Y)$

definition of P :

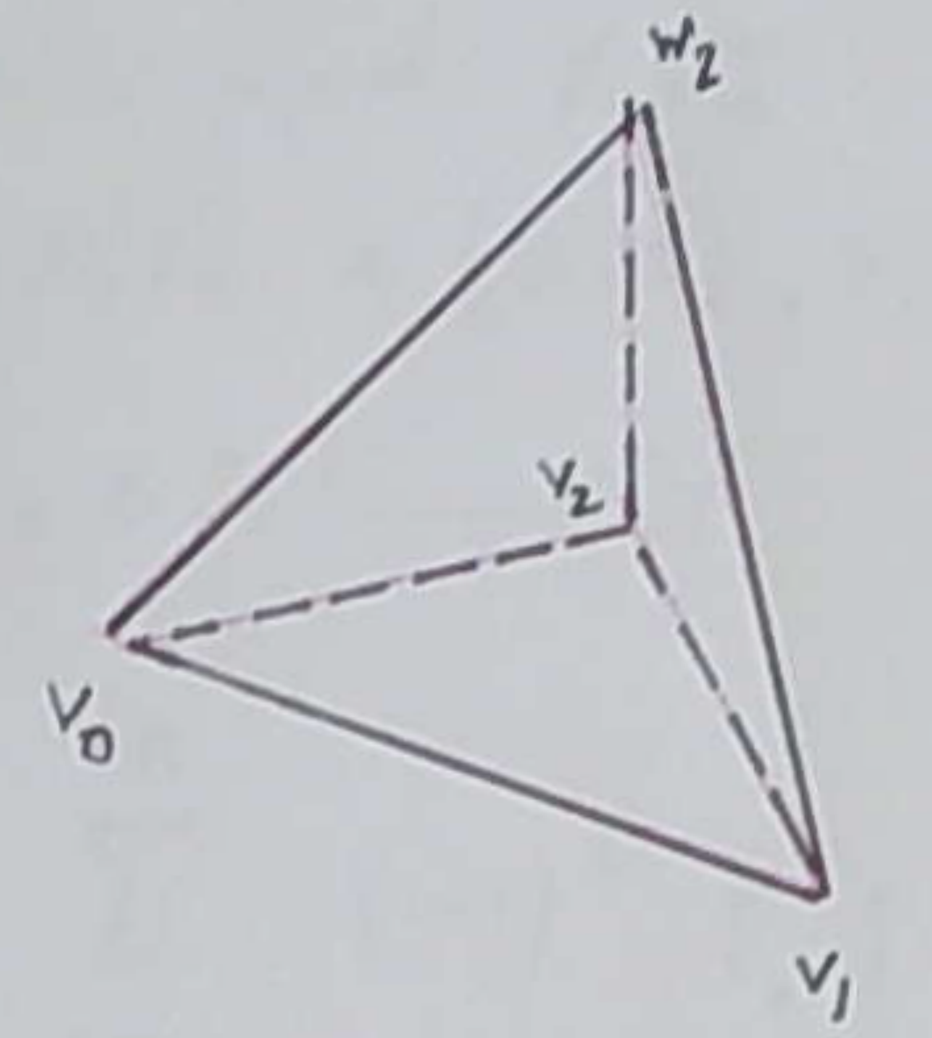
The "prism" $\Delta_n \times I$ can be decomposed into $n+1$ simplices of dimension $n+1$.

$\Delta_n \times 0 = [v_0, \dots v_n]$ $\Delta_n \times 1 = [w_0, \dots, w_n]$

$$\Delta_n \times I = [v_0, w_0, \dots, w_n] + [v_0, v_1, w_1, \dots, w_n]$$

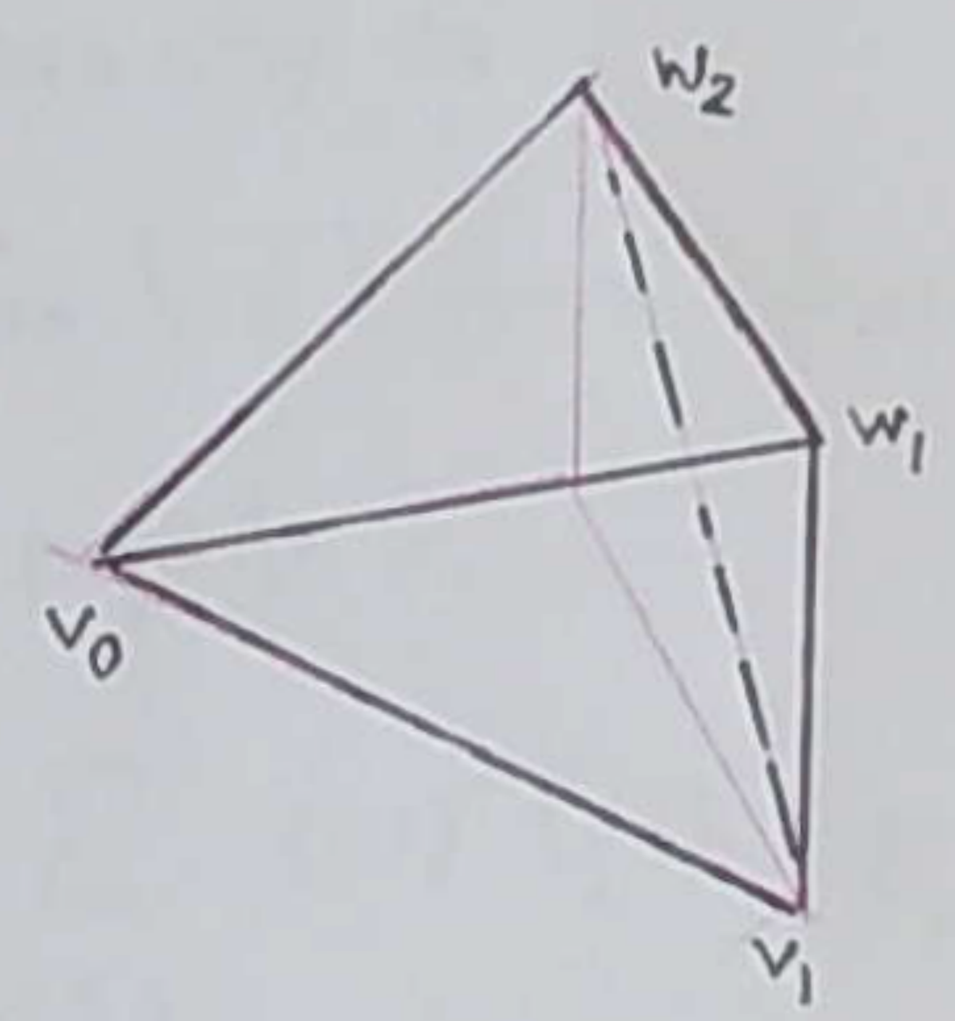
$$+ \dots [v_0, v_1, \dots, v_n, w_n] \quad (\text{in } S_{n+1}(\Delta_n \times I))$$

Example: $\Delta_2 \times I$ subdivides into three 3-simplices



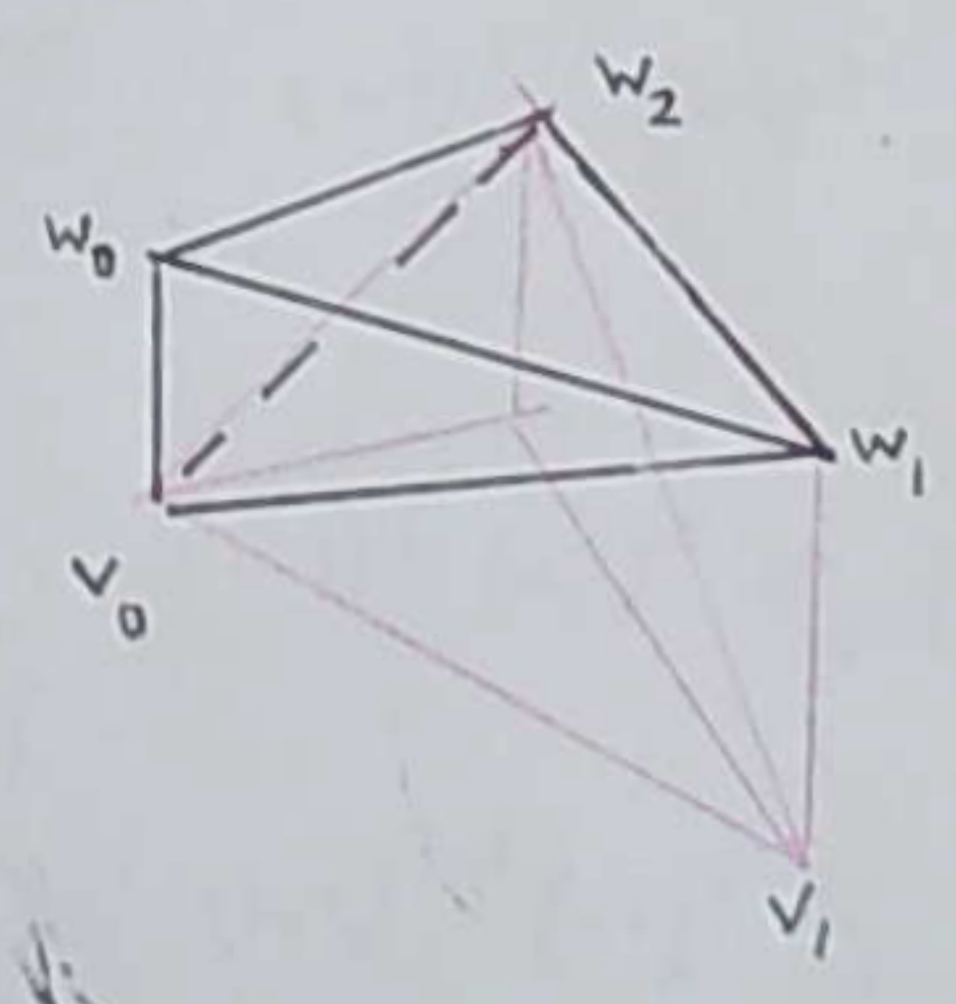
$[v_0, v_1, v_2, w_2]$

$+$



$[v_0, v_1, w_1, w_2]$

$+$



$[v_0, w_0, w_1, w_2]$

Given $F : X \times I \to Y$

$\sigma : \Delta_n \to X$ singular simplex

Def. $\sigma_I : \Delta_n \times I \to X \times I$

$\sigma_I(z,t) = (\sigma(z), t)$

Let

$F_\sigma = F \circ \sigma_I : \Delta_n \times I \to Y$

$$P\sigma = \sum_{i=0}^{n} (-1)^i F_\sigma[v_0, \dots, v_i, w_i, \dots w_n] \in S_{n+1}(Y)$$

(extend by linearity to:

$$P : S_n(X) \to S_{n+1}(Y)$$

$\partial\sigma = \sum_{j=0}^{n} (-1)^j \sigma[v_0 \dots, \hat{v}_j, \dots v_n]$

chain homotopy ~~property~~

$$\partial(P\sigma) = \sum_{j \leq i} (-1)^j (-1)^i F_\sigma[v_0, \dots, \hat{v}_j \dots v_i, w_i, \dots, w_n]$$

$$+ \sum_{j \geq i} (-1)^{j+1} (-1)^i F_\sigma[v_0, \dots, v_i, w_i, \dots, \hat{w}_j, \dots w_n]$$

$P(\partial\sigma)$

$$= \sum_{i<j} (-1)^i (-1)^j F_\sigma[v_0 \dots, v_i, w_i, \dots \hat{w}_j, \dots w_n]$$

$$+ \sum_{j<i} (-1)^{i-1} (-1)^j F_\sigma[v_0, \dots \hat{v}_j \dots, v_i, w_i, \dots w_n]$$

The terms w/ $i \neq j$ cancel in $\partial(P\sigma) + P(\partial\sigma)$

Terms w/ $i=j$

$$\sum_{i=0}^{n} F_\sigma[v_0,\dots,\hat{v}_i, w_i,\dots,w_n] - \sum_{i=0}^{n} F_\sigma[v_0,\dots,v_i,\hat{w}_i,\dots,w_n]$$

$$= F_\sigma[\cancel{v_0}\, w_0, w_1,\dots,w_n] - F_\sigma[v_0,v_1,\dots,v_n]$$

(after cancellation) $\cancel{g}_\# \sigma[a_0,\dots,a_n] - f_\# \sigma[a_0,\dots,a_n]$ $\qquad \Delta_n = [a_0,\dots,a_n]$

$n=2$

$$F_\sigma[\hat{v}_0, w_0, w_1, w_2] + F_\sigma[\cancel{v_0}, \hat{v}_1, w_1, w_2] + F_\sigma[v_0, \cancel{v_1}, \hat{v}_2, w_2]$$

$$-\left(F_\sigma[v_0, \cancel{\hat{w}_0}, w_1, w_2] + F_\sigma[v_0, \cancel{v_1}, \hat{w}_1, w_2] + F_\sigma[v_0, v_1, v_2, \hat{w}_2]\right)$$

$$= F_\sigma[w_0, w_1, w_2] - F_\sigma[v_0, v_1, v_2].$$

This shows $\partial(P\sigma) + P(\partial\sigma) = g_\#\sigma - f_\#\sigma$ (chain homotopy)

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Corollaries

① $f: X \to Y$ htopy equivalence $\Rightarrow f_\#: H_n(X) \to H_n(Y)$ is iso

② $A \subset X$ deformation retract of X
$i: A \to X$ inclusion. Then $i_\#: H_n(A) \to H_n(X)$ is iso

Q) What if $A \subset X$ is a retract of X
(but not a def. retract)