Example $A \subset X$ is a retract of X $\quad i: A \to X$ inclusion

$\exists \quad g: X \to A \quad$ s.t. $g \circ i = id_A$

$$H_n(A) \xrightarrow{i_\#} H_n(X) \xrightarrow{g_\#} H_n(A)$$

$$g_\# \circ i_\# = (g \circ i)_\# = \underbrace{(id_A)_\#}_{\text{id on } H_n(A)}.$$

hence $i_\#$ is MONO (injective).

Claim $\quad H_n(X) = \underbrace{i_\# H_n(A)}_{G_1} \oplus \underbrace{\ker g_\#}_{G_2}$

Pf. $\quad$ Let $\gamma \in H_n(X)$

$$\gamma = \underbrace{i_\# g_\# \gamma}_{G_1} + \underbrace{(\gamma - i_\# g_\# \gamma)}_{G_2}$$

If $\quad \alpha \in G_1 \cap G_2$

$\alpha = i_\# \beta$ and $g_\# \alpha = 0$ $\quad$ so $g_\# i_\# \beta = 0$, so $\beta = 0$, so $\alpha = 0$

the case $A = \{x_0\}$ connects w/ $\tilde{H}_n(X)$

In fact $\quad \tilde{H}_n(X) = \underbrace{H_n(X, x_0)}_{\text{relative homology.}}$

Relative homology & the long exact sequence

$A \subset X$ subspace $\qquad S_n(A) \subset S_n(X)$ subgroup

Def $\quad S_n(X,A) = S_n(X) / S_n(A)$ (quotient gp.)

"relative chains"

$\partial : S_{n+1}(A) \longrightarrow S_n(A)$ so induces (via quotients)

$$\partial : S_{n+1}(X,A) \longrightarrow S_n(X,A)$$

$Z_n(X,A) \subset S_n(X,A)$ "relative n-cycles"

$c \in S_n(X)$ s.t. $\partial c \in S_{n-1}(A)$

$B_n(X,A) \subset S_n(X,A)$.

$c \in S_n(X)$ s.t. $c = \partial d + e$ $\qquad d \in S_{n+1}(X)$, $e \in S_n(A)$

(in part. $c \in Z_n(X,A)$)

relative n^{th} homology gp. :

$$H_n(X,A) = Z_n(X,A) / B_n(X,A)$$

$$S_n(A) \xrightarrow{i_\#} S_n(X) \xrightarrow{\pi} S_n(X,A) = S_n(X)/S_n(A)$$

(injective) $\qquad$ (surjective)

$$0 \to S_n(A) \xrightarrow{i_\#} S_n(X) \xrightarrow{\pi} S_n(X,A) \to 0$$

(short) exact sequence $\qquad \operatorname{im} i_\# = \ker \pi$

From this one builds the long exact seq. in homology

$$\cdots \to H_n(A) \xrightarrow{i_\#} H_n(X) \xrightarrow{\pi_\#} H_n(X,A) \xrightarrow{\Delta} H_{n-1}(A) \to \cdots$$

Δ : "connection homomorphism"

Homological algebra

C, D, E : chain complexes (C_n, ∂) $n \geq 0$, (D_n, ∂) $n \geq 0$, (E_n, ∂) $n \geq 0$

Suppose we have a short exact seq. of chain maps.

$$0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0$$

$$\begin{tikzcd}
0 \arrow[r] & C_n \arrow[r, "f"] \arrow[d, "\partial"'] & \overset{w}{D_n} \arrow[r, "g"] \arrow[d, "\partial"'] & \overset{z}{E_n} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & \overset{u}{C_{n-1}} \arrow[r, "f"] \arrow[d, "\partial"'] & \overset{\partial w}{D_{n-1}} \arrow[r, "g"] \arrow[d, "\partial"] & E_{n-1} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & \underset{\partial u}{C_{n-2}} \arrow[r, "f"] & D_{n-2} \arrow[r, "g"] & E_{n-2} \arrow[r] & 0
\end{tikzcd}$$

(short exact at each level)

connecting homomorphism $\Delta : H_n(E) \longrightarrow H_{n-1}(C)$

Let $z \in Z_n(E_n)$ $\partial z = 0 \in E_{n-1}$

$g : D_n \to E_n$ epi

so $\exists$ $w \in D_n$ s.t. $g(w) = z$

$g(\partial w) = \partial(g w) = \partial z = 0$ (exactness) $\ker g|_{D_{n-1}} = \operatorname{im} f|_{C_{n-1}}$

$\partial w = f(u)$, $u \in C_{n-1}$

$f(\partial u) = \partial(f u) = \partial(\partial w) = 0$

so $\partial u = \ker f|_{C_{n-2}} = \{0\}$ (exactness), so $u \in \cancel{Z}\, Z(C_{n-1})$

Then let $\Delta[z] = [u]$ in $H_{n-1}(C)$

to check

1) well-def : $[u]$ in $H_{n-1}(C)$ depends only on $[z] \in H_n(E)$

2) ~~The~~

Long Sequence of hom. in homology

$$\dots \to H_{n+1}(E) \xrightarrow{\Delta} H_n(C) \xrightarrow{f_\#} H_n(D) \xrightarrow{g_\#} H_n(E) \xrightarrow{\Delta} H_{n-1}(C) \to \dots$$

2) This sequence is <u>exact</u> at each map

<u>Proof.</u> see [Hatcher]

<u>back to relative homology</u>

$$\dots \to H_{n+1}(X,A) \xrightarrow{\Delta} H_n(A) \xrightarrow{i_\#} H_n(X) \xrightarrow{j_\#} H_n(X,A) \xrightarrow{\Delta} H_{n-1}(A) \to \dots$$

Let $z \in \cancel{S_n}\ Z_n(X,A) \subset S_n(X,A)$

$z \in S_n(X)$ $\partial z \in S_{n-1}(A)$ $\quad (\partial(\partial z) = 0$, so $\partial z \in Z_{n-1}(A))$

Consider $[\partial z] \in H_{n-1}(A)$

(not zero nec., since $z \in S_n(X)$, not in $S_n(A)$)

$$\Delta[z] = [\partial z] \text{ in } H_{n-1}(A) \qquad (\Delta \leftrightarrow \partial)$$

Q $\qquad H_n(X, x_0) \approx \tilde{H}_n(X)$

more gen'lly: $\quad H_n(X,A) \approx H_n(X/A)$

under certain conditions on the pair (X,A)