ALGEBRAIC TOPOLOGY II: Alternative proofs.

1. Homotopy groups of products and wedges. (following [Hilton])

For products, we have (suppressing basepoints x_0, y_0 in the notation):

$$\pi_n(X \times Y) \approx \pi_n(X) \oplus \pi_n(Y).$$

This is easy to see: we have injections (meaning: maps induced by inclusion) $i_X : \pi_n(X) \to \pi_n(X \times Y), i_Y : \pi_n(Y) \to \pi_n(X \times Y)$, which are in fact mono: given $g : (D^n, \partial D^n) \to (X, x_0)$, if $i_X[g]_X = [(g, y_0)]_{(X \times Y)} = 0$, compose the homotopy in the product with the projection $p_X : X \times Y \to X$ to conclude $[g]_X = 0$. So define a hom from $\pi_n(X \times Y)$ to the direct sum by:

$$\varphi : [f] \mapsto [g]_X + [h]_Y, \text{ if } f : (D^n, \partial D^n) \to (X \times Y, (x_0, y_0)), \quad f(x) = (g(x), h(x)),$$

in other words, $\varphi = p_X \oplus p_Y$. Then φ is *epi*, since $\varphi(i_X[g]_X + i_Y[h]_Y) = [g]_X + [h]_Y$. And it is also *mono*, since $[g]_X + [h]_Y = 0$ gives homotopies g_t from g to x_0, h_t from h to y_0, and thus also (g_t, h_t) from f to (x_0, y_0).

Now consider $X \vee Y$ where X and Y have a single common point z_0. We regard this as a subspace of the product:

$$X \vee Y \sim X \times \{z_0\} \cup \{z_0\} \times Y \subset X \times Y.$$

Consider the injections (of based spaces and maps):

$$j_X : \pi_n(X) \to \pi_n(X \vee Y), \quad j_Y : \pi_n(Y) \to \pi_n(X \vee Y), \quad k : \pi_n(X \vee Y) \to \pi_n(X \times Y).$$

Then $k \circ j_X = i_X$ and $k \circ j_Y = i_Y$, and since i_X, i_Y are mono, so are j_X, j_Y. In general k is not mono. Consider also the hom:

$$\tau : \pi_n(X \times Y) \to \pi_n(X \vee Y), \quad \tau(\alpha) = j_X p_X(\alpha) + j_Y p_Y(\alpha).$$

Note $\tau \circ i_X = j_X, \tau \circ i_Y = j_Y$, and also, for any $\alpha \in \pi_n(X \times Y)$:

$$k\tau(\alpha) = k(i_X p_X)(\alpha) + k(j_Y p_Y)(\alpha) = (kj_X)p_X(\alpha) + (kj_Y)p_Y(\alpha) = i_X p_X(\alpha) + i_Y p_Y(\alpha) = \alpha.$$

So τ is a right inverse to k, and thus k is epi, τ is mono, and τ embeds $\pi_n(X \times Y)$ into $\pi_n(X \vee Y)$:

$$\tau\pi_n(X \times Y) = \tau i_X \pi_n(X) \oplus \tau i_Y \pi_n(Y) = j_X \pi_n(X) \oplus j_Y \pi_n(Y) \approx \pi_n(X) \oplus \pi_n(Y).$$

We claim: $\pi_n(X \vee Y) = im(\tau) \oplus ker(k)$. Indeed the fact $k\tau = id$ shows these subgroups intersect at $\{0\}$, and note any $\gamma \in \pi_n(X \vee Y)$ decomposes as:

$$\gamma = \tau k(\gamma) + (\gamma - (\tau k)(\gamma)) \in im(\tau) \oplus ker(k).$$

We conclude:

$$\pi_n(X \vee Y) \approx \pi_n(X) \oplus \pi_n(Y) \oplus ker(k).$$

To identify $ker(k)$, consider the homotopy exact sequence for the pair $(X \times Y, X \vee Y)$:

$$\ldots \xrightarrow{\mu} \pi_{n+1}(X \times Y, X \vee Y) \xrightarrow{\partial} \pi_n(X \vee Y) \xrightarrow{k} \pi_n(X \times Y) \xrightarrow{\mu} \pi_n(X \times Y, X \vee Y) \xrightarrow{\partial} \pi_{n-1}(X \vee Y) \to \ldots$$

We have: k is epi, so μ is the zero map and $ker(\partial) = im(\mu) = 0$, so ∂ is mono and $ker(k) = im(\partial) \approx \pi_{n+1}(X \times Y, X \vee Y)$. We conclude, finally;

$$\pi_n(X \vee Y) \approx \pi_n(X) \oplus \pi_n(Y) \oplus \pi_{n+1}(X \times Y, X \vee Y).$$

Example. Consider the wedge of two n-spheres, $X = S^n_1 \vee S^n_2$. With a standard cell decomposition of S^n (one n-cell, one 0-cell), X has the product cell decomposition: one 0-cell (the basepoint), two n-cells (which combined give the decomposition of the wedge) and one $2n$-cell. Thus the pair $(X, S^n_1 \vee S^n_2)$ is $(n+1)$-connected, its relative homotopy group π_{n+1} vanishes and we have: $\pi_n(X) \approx \pi_n(S^n_1) \oplus \pi_n(S^n_2) \approx \mathbb{Z}^2$.

A similar reasoning applies to give for $\pi_n(X) \approx \mathbb{Z}^N$, if X is a 'bouquet'(!) of N n-spheres. (Since its cell decomposition only has cells with dimensions a multiple of n.)

2. Freudenthal suspension theorem.

Denote by $S : \pi_q(S^n) \to \pi_{q+1}(S^{n+1})$ the suspension homomorphism. (In the sphere case, $S[f]$ is represented by any extension of $f : S^q \to S^n$–regarded as the equators of S^{q+1}, S^{n+1}– to a map $S^{q+1} \to S^{n+1}$ preserving meridians.) Note that in the theorem the equatorial S^q, S^n are chosen once and for all.

Theorem: The suspension hom S is an epimorphism if $n \leq q \leq 2n-1$, an isomorphism if $n \leq q < 2n-1$.

Remark: More generally, the suspension hom $S : \pi_q(X) \to \pi_{q+1}(S(X))$ is epi (resp. iso) in the same range of dimensions, if X is an $(n-1)$-connected CW complex.

Proof (outline) We present essentially the geometric proof described in [Fomenko-Fuchs, p. 121], where you'll find all the helpful pictures. By approximation within the same homotopy class, we may assume the maps and homotopies that occur are *smooth*, which simplifies things a bit. The proof is based on the geometric fact that, in this range of dimensions, preimages of points are *unlinked* submanifolds.

Recall two compact, embedded, disjoint submanifolds $P, Q \subset \mathbb{R}^N$ are *unlinked* if one may find an isotopy of $\mathbb{R}^n$ (for example an orientation-preserving isometry) φ so that $\varphi(P)$ and Q can be separated by a hyperplane. For instance, it is easy to draw linked embeddings of S^1 in $\mathbb{R}^3$. More simply, two Jordan curves in the plane, one inside the other (or a point inside a Jordan curve) are linked (=not unlinked), but become unlinked as submanifolds of $\mathbb{R}^3$.

The following is well-known: two compact embedded disjoint submanifolds $P, Q \subset \mathbb{R}^N$ are unlinked if $dim(P) + dim(Q) < N - 1$. To see this, consider the embeddings $f : P \to R^N, g : Q \to R^N$ and the smooth map $F : P \times Q \to S^{N-1}$ obtained by normalizing the vector $f(p) - g(q)$. Let $n_0 \in S^{N-1}$ be a regular value of F. Due to the dimension condition, this can only mean $F^{-1}(n_0) = \emptyset$. Hence no line in $\mathbb{R}^N$ with direction vector n_0 meets both P and Q. Now move P by a translation φ with direction n_0 sufficiently far (never meeting Q), so that some hyperplane $\mathcal{H}$ normal to n_0 will have $\varphi(P)$ and Q on opposite sides. (In general, the *linking number* of two disjoint, embedded compact oriented submanifolds $P, Q \subset \mathbb{R}^N$ with $dim(P) + dim(Q) = N - 1$ may be defined (following Gauss) as the degree of this map F.)

The suspension homomorphism is surjective if $n \leq q \leq 2n - 1$. This means: given $f : S^{q+1} \to S^{n+1}$, we may deform f by successive homotopies and find a map $g : S^q \to S^n$ so that $f \simeq Sg$ (for the new f). Model $S^{q+1} = \mathbb{R}^{q+1} \cup \{\infty\}$, and let N, S be the north and south poles of S^{n+1}. We may assume N, S are regular values of f (and that $f(\infty) \notin \{N, S\}$), so their preimages are disjoint submanifolds $P, Q \subset \mathbb{R}^{q+1}$ (maybe disconnected), both of dimension $q - n$. Select also neighborhoods U, V of N, S, whose preimages are neighborhoods U_1, V_1 of P, Q.

Now, the dimension condition implies $dim(P)+dim(Q) = 2(q-n) < q$, hence P, Q are unlinked, and we may find an isotopy φ of S^{q+1} so that P, Q are in opposite hemispheres (relative to some equator), and furthermore (by shrinking U, V if needed) so that each of U_1, V_1 is contained in the same hemisphere as the new P, Q (resp.) Now find a rotation (element of $SO(q + 1)$) that moves this equator to the original one, $S^q \subset S^{q+1}$, so that U_1, V_1 are now contained in the northern (resp. southern) hemisphere of S^{q+1}, and map (by the new f) to $U, V \subset S^{n+1}$ (resp.)

From this point on the proof of surjectivity proceeds as in [F-F].

The suspension homomorphism is injective if $n \leq q < 2n - 1$. This means: given $f, g : S^q \to S^n$, if $Sf \simeq Sg$ we have $f \simeq g$. Naturally, if $h_t : S^{q+1} \to S^{n+1}$ is a homotopy from Sf to Sg, we seek to deform h_t to a homotopy of the form $S(f_t)$, where f_t is a homotopy from f to g. Thinking of the h_t as $H : S^{q+1} \times I \to S^{n+1}$, we seek to deform H to another homotopy $H_1 : S^{q+1} \times I \to S^{n+1}$, which on each fiber $t = const$ is the suspension of a map $S^q \to S^n$. As before we assume N, S are regular values of H and consider their preimages P, Q under H, each a submanifold of $R^{q+1} \times R = R^{q+2}$ of dimension $q - n + 1$. And now the unlinking criterion (with separation by a *fixed* hyperplane in R^{q+2}) is:

$$2(q - n + 1) < q + 2 - 1, \text{ or } q < 2n - 1.$$

From this point on, the proof of injectivity proceeds as in [F-F].

Remark: The kernel of the suspension map $S : \pi_{2n-1}(S^n) \to \pi_{2n}(S^{n+1})$ can be described in terms of the *Whitehead product* (see [F=F, p.127 ff.):

$$\alpha \in \pi_m(X, x_0), \beta \in \pi_n(X, x_0) \mapsto [\alpha, \beta] \in \pi_{m+n-1}(X, x_0).$$

The kernel is the cyclic group generated by $[I_n, I_n] \in \pi_{2n-1}(S^n)$, where $I_n \in \pi_n(S^n)$ is the homotopy class of the identity map.

3. Proof of the Hurewicz theorem.

Hurewicz Theorem. Let X be a connected CW complex. Assume X is $(n-1)$-connected, where $n \geq 1$ ($\pi_p(X) = 0, p = 1, \ldots, n-1$.) Then $\tilde{H}_p(X) = 0$ for $p = 1, \ldots, n-1$ (reduced homology) and the Hurewicz homomorphism $h : \pi_n(X) \to \tilde{H}_n(X)$ is an isomorphism.

Proof. (Based on cellular homology.)

1. Reduction. Using $(n-1)$-connectivity, we may replace X by a homotopically equivalent complex with a single 0-cell x_0 and no other cells of dimension $< n$; that is, the $(n-1)$ skeleton $X^{n-1} = \{x_0\}$. This already implies (via cellular homology) the reduced homology vanishes in dimensions $0, \ldots, n-1$.

Additionally, note that $\pi_n(X)$ depends only on the $(n+1)$-skeleton (by cellular approximation of maps) and likewise for $\tilde{H}_n(X)$ (by cellular homology). So we might as well assume $X = X^{n+1}$–there are no cells of dimension $> n+1$– and we do.

This reduces the proof to the following special situation:

$$X^n = \bigvee_{k \in K} S^n_k, \quad X = X^n \sqcup_{l \in L} e^{n+1}_l.$$

That is, the n-skeleton is a wedge of n-spheres and X is obtained from it by attaching $(n+1)$-cells e^{n+1}_l, via attaching maps $\varphi_l : S^n \to X^n$ (with corresponding characteristic maps $\Phi_l : D^{n+1} \to X$, restricting to φ_l on ∂D^{n+1}).

In this particular case, π_n is easy to describe: it is the quotient of the free abelian group $\pi_n(X^n) \approx \bigoplus_{k \in K} \pi_n(S^n_k)$ on K generators (represented by $i_{k*}[id_n]$, where $[id_n] \in \pi_n(S^n)$ is the class of the identity map of S^n and $i_k : S^n \to X^n$ is the inclusion map of the k^{th} sphere of the wedge) by the subgroup of $\pi_n(X^n)$ generated by the attachment maps $\varphi_l : S^n \to X^n$ of the $(n+1)$-cells:

$$\pi_n(X) = \pi_n(X^n)/\langle\{[\varphi_l]_{X^n}; l \in L\}\rangle$$

(See Example 4.29 in [Hatcher], which unfortunately ultimately relies on the 'homotopy excision theorem'. Alternatively, use the 'homotopy addition theorem', as described in sections 11.1, 11.2, 11.3 of [Fomenko-Fuchs].)

2. Reminder of cellular homology facts. To understand the homology side of things, we appeal to cellular homology. Recall the cellular chain complex $(C_n(X), d_n)_{n \geq 0}$ of a CW complex is given (in terms of singular homology) by:

$$C_n(X) = H_n(X^n, X^{n-1}), \quad d_n = j_* \circ \partial_n : C_n(X) \to C_{n-1}(X) = H_{n-1}(X^{n-1}, X^{n-2}),$$

where $\partial_n : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1})$ is the connection homomorphism of the homology exact sequence of the pair (X^n, X^{n-1}) and $j_* : H_{n-1}(X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$ is induced by inclusion of pairs. Recall $C_n(X)$ is free abelian, with generators in bijective correspondence with the set of n-cells of X.

In our case, $X^{n-1} = \{x_0\}$, so $H_n(X^n, X^{n-1}) = \tilde{H}(X^n)$, and we may identify j_* with the identity in $\tilde{H}_n(X^n)$ and d_{n+1} with $\partial_{n+1} : C_{n+1}(X) \to \tilde{H}_n(X)$ (since $n \geq 2$, homology and reduced homology coincide in these dimensions.) Also, since $C_{n-1}(X) = 0$ all chains in $C_n(X)$ are cycles, and thus:

$$H_n^{cell}(X) = C_n(X)/im(d_{n+1}).$$

(And $H_n^{cell}(X) \approx H_n(X)$.)

3. The Hurewicz map of a wedge of spheres of the same dimension. Here we consider $h : \pi_n(X^n) \to \tilde{H}_n(X^n) = \tilde{H}_n(X)$. Note that $C_{n+1}(X^n) = 0$, since X^n is a wedge of n-spheres and has no $(n+1)$-cells. So we have:

$$h : \pi_n(X^n) \to C_n(X^n) = \tilde{H}_n(X^n) \approx \bigoplus_{k \in K} \tilde{H}_n(S_k^n),$$

(reduced singular homology on the right) where both groups are isomorphic to $\oplus_K \mathbb{Z}$.

As we recalled above, $\pi_n(X^n)$ has a basis $\{i_{k_*}[id_n] = [i_k]_{X^n}\}_{k \in K}$, where $i_k : S^n \to X^n$ is inclusion as the k^{th} sphere. What is the image of the k^{th} basis element under h? By definition, it is $(i_k)_\#(s_n) \in \tilde{H}_n(X^n)$ (induced hom in homology), where $s_n \in H_n(S^n)$ is a fixed generator:

$$h([i_k]_{X^n}) = (i_k)_\#(s_n).$$

Note $(i_k)_\# : \tilde{H}_n(S^n) \to \tilde{H}_n(X^n)$ is inclusion of the k^{th} summand into a direct sum. Hence $(i_k)_\#(s_n)$ is a basis element of $\tilde{H}_n(X^n)$.This shows the map $h : \pi_n(X^n) \to \tilde{H}_n(X^n)$ is an isomorphism, which of course is the Hurewicz theorem for a wedge of n-spheres.

4. Conclusion of the proof. Given the above descriptions of $\pi_n(X)$ and $\tilde{H}_n(X)$ as quotient groups, all that is left to show is that the Hurewicz homomorphism h satisfies the mapping condition of subgroups (of $\pi_n(X)$ and $\tilde{H}_n(X)$):

$$h(\langle [\varphi_l]_{X^n}; l \in L \rangle) = im(d_{n+1}),$$

where on the left we have the subgroup of $\pi_n(X)$ generated by attachment maps $\varphi_l : S^n \to X^n$ of the cells e_l^{n+1}, and on the right the image of the connection homomorphism $d_{n+1} = j_* \partial_{n+1} : C_{n+1}(X) = H_{n+1}(X^{n+1}, X^n) \to \tilde{H}_n(X^n) = C_n(X)$ (composition of a connecting hom ∂_{n+1} in an exact sequence of pairs and an injection operator $j_* : H_n(X^n) \to H_n(X^n, X^{n-1}) \approx \tilde{H}_n(X^n) = C_n(X)$).

Note that the attaching map φ_l of e_l^{n+1} extends to the characteristic map $\Phi_l : (D^{n+1}, S^n) \to (X, X^n) = (X^{n+1}, X^n)$, inducing in homology the hom

$(\Phi_l)_\# : H_{n+1}(D^{n+1}, S^n) \to H_{n+1}(X^{n+1}, X^n) = C_{n+1}(X)$. By naturality of connection homomorphisms, we have the commutative diagram:

$$\begin{array}{ccc} H_{n+1}(D^{n+1}, S^n) & \xrightarrow{\partial_{n+1}} & H_n(S^n) \\ \downarrow{\scriptstyle \Phi_{l\#}} & & \downarrow{\scriptstyle \varphi_{l\#}} \\ C_{n+1}(X) & \xrightarrow{\partial_{n+1}} & C_n(X) \end{array}$$

Note ∂_{n+1} (top row) is an isomorphism, and $(\partial_{n+1})^{-1}$maps the generator s_n of $H_n(S_n)$ used to define h to a generator r_{n+1} of the infinite cyclic abelian group $H_{n+1}(D^{n+1}, S^n)$. Commutativity of the diagram implies:

$$h([\varphi_l]_X) = (\varphi_l)_\#(s_n) = \partial_{n+1}(\Phi_l)_\#(r_{n+1}),$$

and therefore $h([\varphi_l]_X) \in im(d_{n+1})$, since we're identifying ∂_{n+1} (with image in $H_n(X^n)$) and d_{n+1} (with image in $\tilde{H}_n(X_n) \approx C_n(X)$).

Now recall $C_{n+1}(X) = H_{n+1}(X^{n+1}, X^n)$, and since (X^{n+1}, X^n) is a CW pair, we have $H_{n+1}(X^{n+1}, X_n) \approx \tilde{H}_{n+1}(X^{n+1}/X^n)$, while $X^{n+1}/X^n = \bigvee_{l \in L} S_l^{n+1}$, a wedge of spheres with $\tilde{H}_{n+1}$ given by the direct sum of individual $\tilde{H}_{n+1}$'s. Thus the $(\Phi_l)_\#(r_{n+1})$ form a basis of $C_{n+1}(X)$, and we have in fact established the equality of the subgroups $h(\langle[\varphi_l]_{X^n}; l \in L\rangle)$ and $im(d_{n+1})$, concluding the proof.

Sources: This proof follows the idea outlined in [Hatcher, section 4.2], which emphasizes the relative case, but I've included more detail. The main reason to choose a proof based on cellular homology is that it ties in with the existence/uniqueness (up to homotopy type) of Eilenberg-MacLane spaces.

A reasonably concise and understandable proof based on simplicial homology is found in S-T Hu, *Homotopy Theory* (1959), Theorem 4.4 (Chapter V, section 4).

Classification of vector bundles.

1. Vector bundle morphisms and isomorphism.

Let ξ, η be k, l-vector bundles over base spaces $B(\xi), B(\eta)$. A vector bundle morphism is a pair (u, f) of continuous maps, where $u : E(\xi) \to E(\eta)$ preserves fibers and is linear in each fiber, and $f : B(\xi) \to B(\eta)$ is the induced map of base spaces.

If ξ, η have the same base space B, u is a linear isomorphism on fibers (so $k = l$) and the induced map $f = id_B$, we say ξ and η are (strongly) isomorphic (Notation: $\xi \approx \eta$). It is easy to show [M-S Lemma 3.1] this implies $u : E(\xi) \to E(\eta)$ is a homeomorphism.

Let $f : B \to X$, η a k-vector bundle over X. Recall the pullback $f^*\eta$ has total space and projection:

$$E(f^*\eta) = \{(b, v) \in B \times E(\eta); f(b) = p_\eta(v) \in X\}, \quad p(b, v) = b.$$

We have the following easy but useful fact:

Proposition. Suppose $(u, f) : \xi \to \eta$ is a vector bundle morphism and an isomorphism on each fiber. Then $\xi \approx f^*\eta$ (as vector bundles over B.)

For the *proof* we define $h : E(\xi) \to E(f^*\eta)$ via: $h(e) = (p_\xi(e), u(e))$ (note $p_\eta(u(e)) = f(p_\xi(e))$, so this makes sense.) Then h is continuous and maps each fiber $V_b(\xi)$ isomorphically onto $V_b(f^*\eta)$ (since on each fiber $V_b(\xi)$, h coincides with u.)

2. Canonical and universal k-vector bundles.

We denote by γ_k^n the canonical k-plane bundle over the real Grassmannian $G_k(R^n)$ $(n \geq k)$, with total space and projection:

$$E(\gamma_k^n) = \{(X, v) \in G_k(R^n) \times R^n; v \in X.\}, \quad p(X, v) = X.\}$$

The universal k-vector bundle γ_k has as base space the $G_k(R^\infty)$, the infinite increasing union of the $G_k(R^n)$, an inifnite-dimensional CW complex given the 'weak topology', with total space:

$$E(\gamma_k) = \{(X, v) \in G_k(R^\infty) \times R^\infty; v \in X\}, \quad p(X, v) = X.$$

Note $E(\gamma_k)$ is the infinite increasing union of the $E(\gamma_k^n)$, with compatible projection maps.

The canonical bundle γ_k^n over $G_k(R^n)$ is associated with an equally canonical $(n-k)$-vector bundle over the same base, its orthogonal complement $\gamma_k^{n\perp}$, with total space and projection:

$$E(\gamma_k^{m\perp}) = \{(X, v) \in G_k(R^n) \times R^n; v \in X^\perp\}, \quad q(X, v) = X.$$

Note that their Whitney sum is the trivial n-bundle over $G_k(R^n)$:

$$\gamma_k^n \oplus \gamma_k^{n\perp} = \epsilon^n := G_k(R^n) \times R^n.$$

This implies that if ξ is a k-vector bundle over B and $\xi \approx f^*\gamma_k^n$, for some n and some $f : B \to G_k(R^n)$, then ξ admits a 'complement', a $n-k$-vector bundle η over B so that:

$$\xi \oplus \eta \approx \epsilon_B^n := B \times R^n,$$

the trivial n-bundle over B. (Just let $\eta = f^*(\gamma_k^{n\perp})$.)

3. Existence theorem.

Theorem 1. Let ξ be a k-plane bundle over a compact manifold B. Then there exists n and $f : B \to G_k(R^n)$ so that $\xi \approx f^*\gamma_k^n$.

Proof. See [M-S, Lemma 5.3] (extended to paracompact base spaces in Theorem 5.6, with maps to $G_k(R^\infty)$.) In the compact case, we have an open cover of B by finitely many (say N) open sets, over each of which ξ is trivial. The proof gives a map $B \to G_k(R^{kN})$, using partitions of unity in the same way as the proof that (compact) manifolds embed in euclidean spaces of sufficiently large dimension.

4. Homotopy implies isomorphism.

Theorem 2. Suppose $f_0, f_1 : B \to G_k(R^n)$ are homotopic maps (B compact.) Then the pullback bundles are isomorphic: $f_0^*\gamma_k^n \approx f_1^*\gamma_k^n$.

Proof. (Adapted from [Benedetti, p. 100].) Consider the simple linear algebra fact: suppose we have two direct sum decompositions

$$R^n = V' \oplus V = V'' \oplus V.$$

Then we have a canonical isomorphism $\phi : V' \to V'', \phi(v') = v''$ where $v' = v'' + v$ (unique decomposition.)

Let $F : B \times [0,1] \to G_k(R^n)$ be the homotopy from f_0 to f_1, and consider $F^*(\gamma_k^n)$, a k-vector bundle over $B \times [0,1]$. Denote by $V_{p,t}$ its fiber over (p,t), a k-dimensional subspace of R^n depending continuously on (p,t). Observe the following. For any given $t \in [0,1]$, we have:

$$V_{p,t} \cap V_{p,0}^{\perp} = \{0\}, \forall p \in B \Leftrightarrow R^n = V_{p,t} \oplus V_{p,0}^{\perp}, \forall p \in B \Leftrightarrow f_t^*(\gamma_k^n) \approx f_0^*(\gamma_k^n).$$

Indeed it suffices to consider that the linear algebra fact implies the existence of a continuous field of linear isomorphisms:

$$\phi_p : V_{p,t} \to V_{p,0}, \quad p \in B,$$

that is, of a bundle isomorphism $\phi : f_t^*(\gamma_k^n) \approx f_0^*(\gamma_k^n)$. Clearly the set of $t \in [0,1]$ such that $V_{p,t} \cap V_{p,0}^{\perp} = \{0\}, \forall p \in B$ is open in [0,1]. (This condition is equivalent to the orthogonal projection in R^n mapping $V_{p,t}^{\perp}$ isomorphically onto $V_{p,0}^{\perp}$.)

Claim. $\exists \varepsilon > 0$ such that $\forall 0 \leq t \leq \epsilon, \forall p \in B, V_{p,t} \oplus V_{p,0}^{\perp} = R^n$.

Proof. Otherwise we have a sequence $(p_n, t_n) \to (p_0, 0)$ in $B \times [0,1]$, such that $\dim(V_{p_n,t_n} \cap V_{p_0,0}^{\perp}) > 0$, which in the limit gives $\dim(V_{p_0,0} \cap V_{p_0,0}^{\perp}) > 0$, contradiction.

Thus if we consider the set:

$$\mathcal{G} = \{\varepsilon \in [0,t]; V_{p,t} \cap V_{p,0}^{\perp} = \{0\}, \forall p \in B, \forall 0 \leq t \leq \varepsilon\} = \{t \in [0,1]; f_t^*\gamma_k^n \approx f_0^*\gamma_k^n, \forall 0 \leq t \leq \varepsilon\},$$

we see that $\mathcal{G}$ is a an interval $[0, \epsilon_0)$, open on the right unless $\epsilon_0 = \sup \mathcal{G} = 1$.

But in fact $\epsilon_0 \in \mathcal{G}$: let $t_m \in \mathcal{G}, t_m \uparrow \epsilon_0$. Since $R^n = V_{p\epsilon_0} \oplus V_{p\epsilon_0}^{\perp} \forall p$, we have $V_{pt_m} \cap V_{\epsilon_0 p} = \{0\}^{\perp} \forall p$ for m sufficiently large (due to openness of the condition), hence $f_{t_m}^*\gamma_k^n \approx f_{\epsilon_0}^*\gamma_k^n$ for m large, and since $t_m \in \mathcal{G}$ also $f_0^*\gamma_k^n \approx f_{\epsilon_0}^*\gamma_k^n$, so $\epsilon_0 \in \mathcal{G}$. This implies $\epsilon_0 = 1$, or $f_0^*\gamma_k^n \approx f_1^*\gamma_k^n$, as we wished to show.

5. Gauss maps. [Husemoller p.31.] Definition: a Gauss map to R^m for a k-vector bundle ξ is a continuous map $g : E(\xi) \to R^m$ which is a linear monomorphism on each fiber of ξ (so $m \geq k$.)

For example, $q : E(\gamma_k^n) \to R^n, q(X, v) = v$ is a Gauss map. If $(u, f) : \xi^k \to \gamma_k^n$ is a bundle morphism which is isomorphic on fibers, the composition $q \circ u : E(\xi) \to R^n$ is a Gauss map.

Conversely, if ξ is a k-vector bundle $p : E(\xi) \to B$ and $g : E(\xi) \to R^m$ is a Gauss map for ξ, there exists a bundle morphism $(u, f) : \xi \to \gamma_k^m$ such that $q \circ u = g$.

To see this, for $b \in B$ let $f(b) = g(p^{-1}b)$, the image under g of the fiber of ξ over b, a k-dimensional subspace of R^m and hence a point of $G_k(R^m)$; and for $e \in E(\xi)$, let $u(e) = (f(p(e)), g(e)) \in E(\gamma_k^m)$. Using local trivializations, one sees that u and f are continuous and u is isomorphic on fibers. Thus we have the following simple but useful observation, for an arbitrary k-vector bundle $\xi(E, p, B)$:

$$[\exists f : B \to G_k(R^m), \xi \approx f^*(\gamma_k^m)] \Leftrightarrow [\exists g : E \to R^m \text{ Gauss map}].$$

In particular, it follows from Theorem 1 that any vector bundle over a compact (or paracompact) base admits a Gauss map.

6. The even-odd trick. [Husemoller p. 33, M-S p. 67, Thm 5.7]

To finish the classification theorem, it remains to prove that isomorphism implies homotopy: if $f_0, f_1 : B \to G_k(R^n)$ yield isomorphic vector bundles under pullback: $f_0^*\gamma_k^n \approx f_1^*\gamma_k^n$, then they are homotopic: $f_0 \simeq f_1$ as maps to $G_k(R^n)$ (that is, with the homotopy taking values in $G_k(R^n)$). Unfortunately this is not what is proved, and here is the problem: say $f_0^*\gamma_k^n \approx f_1^*\gamma_k^n \approx \xi$ a k-vector bundle over B, the isomorphisms being given by $u_0, u_1 : E(\xi) \to E(\gamma_k^n)$, inducing f_0, f_1. It would be enough to produce a homotopy between the corresponding Gauss maps $g_0 = q \circ u_0, g_1 = q \circ u_1$, an easier problem since we may try the linear homotopy in R^n:

$$g_t(e) = (1 - t)g_0(e) + tg_1(e) \in R^n, \quad e \in E(\xi), t \in [0, 1].$$

Unfortunately there is no way to guarantee this always gives a nonzero vector if $e \neq 0$; that is, that each g_t is itself a Gauss map.

Thus a tricky detour is necessary, which in the end results in a homotopy from f_0 to f_1, but taking values in $G_k(R^{2n})$, not $G_k(R^n)$.

Consider the 'even and odd subspaces' of R^∞:

$$R^{ev} = \{x \in R^\infty; x_{2i+1} = 0 \forall i \geq 0\}; \quad R^{od} = \{x \in R^\infty; x_{2i} = 0 \forall i \geq 0\}.$$

For each $t \in [0, 1]$, consider the linear embeddings:

$$k_t^e : R^n \to R^{2n}, k_t^o : R^n \to R^{2n} :$$

$$k_t^e(x_0, x_1, \ldots, x_{n-1}) = (1-t)(x_0, x_1, \ldots, x_{n-1}, 0, \ldots, 0) + t(x_0, 0, x_1, 0, \ldots, x_{n-1}, 0).$$

$$k_t^o(x_0, x_1, \ldots, x_{n-1}) = (1-t)(x_0, x_1, \ldots, x_{n-1}, 0, \ldots, 0) + t(0, x_0, 0, x_1, \ldots, 0, x_{n-1}).$$

We see that each k_t^e, k_t^o ($t \in [0, 1]$) is a linear embedding and:

(1) $k_0^e = k_0^o$ is the standard inclusion $R^n \to R^{2n}$ (set the last n coordinates equal to 0).

(2) $k_1^e(R^n) = R^{2n} \cap R^{ev}$, $k_1^0(R^n) = R^{2n} \cap R^{odd}$.

(3) Denote by $q_n : E(\gamma_k^n) \to R^n, q_{2n} : E(\gamma_k^{2n}) \to R^{2n}$ the canonical Gauss maps. Then $k_1^e \circ q_n, k_1^o \circ q_n$ are Gauss maps for γ_k^n, taking values in R^{2n}. Thus, as seen above, there exist vector bundle morphisms (injective on fibers):

$$(u^e, f^e) : \gamma_k^n \to \gamma_k^{2n}, \quad (u^o, f^o) : \gamma_k^n \to \gamma_k^{2n},$$

such that:

$$k_1^e \circ q_n = q_{2n} \circ u^e, \quad k_1^o \circ q_n = q_{2n} \circ u^o.$$

(4) $f^e, f^o : G_k(R^n) \to G_k(R^{2n})$ are homotopic to the standard inclusion maps $j : G_k(R^n) \to G_k(R^{2n})$. Namely,

$$X \mapsto k_t^e(X) \in G_k(R^{2n}), X \in G_k(R^n) \text{ joins } j \text{ at } t = 0 \text{ to} f^e \text{ at } t = 1;$$

$$X \mapsto k_t^o(X) \in G_k(R^{2n}), X \in G_k(R^n) \text{ joins } j \text{ at } t = 0 \text{ to} f^o \text{ at } t = 1.$$

7. From isomorphism to homotopy.

Theorem 3. Let $f_0, f_1 : B \to G_k(R^n)$ such that $f_0^*(\gamma_k^n) \approx f_1^*(\gamma_k^n)$. Then $j \circ f_0 \simeq j \circ f_1$ (homotopic), where $j : G_k(R^n) \to G_k(R^{2n})$ is the standard inclusion.

Proof. (cp. [Husemoller, p. 35].) By hypothesis there exists a k-vector bundle ξ over B and bundle morphisms (isomorphic on fibers):

$$(u_0, f_0) : \xi \to \gamma_k^n, \quad (u_1, f_1) : \xi \to \gamma_k^n$$

and Gauss maps:

$$h_0 = q_n \circ u_0 : E(\xi) \to R^n, \quad h_1 = q_n \circ u_1 : E(\xi) \to R^n.$$

Composing with the maps obtained in the previous subsection, we have:

$$(u^e \circ u_0, f^e \circ f_0) : \xi \to \gamma_k^{2n} \text{ with Gauss map } k_1^e \circ h_0 : E(\xi) \to R^{2n} \cap R^{ev},$$

$$(u^o \circ u_1, f^o \circ f_1) : \xi \to \gamma_k^{2n} \text{ with Gauss map } k_1^o \circ h_1 : E(\xi) \to R^{2n} \cap R^{odd}.$$

Define the map $h_t : E(\xi) \to R^{2n}$, $t \in [0, 1]$:

$$h_t(e) = (1 - t)(k_1^e \circ h_0)(e) + t(k_1^o \circ h_1)(e), \quad e \in E(\xi).$$

Then h_t is a Gauss map for ξ, for each t: in each fiber $V_b(\xi)$, $k_1^e \circ h_0 : V_b(\xi) \to R^{ev}, k_1^e \circ h_1 : V_b(\xi) \to R^{odd}$ are both injective, taking values in subspaces intersecting only at 0; thus h_t is also injective.

This implies there exists a continuous one-parameter family of bundle morphisms:

$$(w_t, \phi_t) : \xi \to \gamma_k^{2n},$$

where $\phi_t : B \to G_k(R^{2n})$ is a homotopy $f^e \circ f_0 \simeq f^o \circ f_1$.

Now recall (from point (4) above): $j \circ f_0 \simeq f^e \circ f_0, j \circ f_1 \simeq f^o \circ f_1$. Thus $j \circ f_0 \simeq j \circ f_1$, as we wished to show.

8. Summary. Theorems 1,2,3 may be summarized as the classification theorem:

Any real k-vector bundle ξ over a compact (or paracompact) base space B is the pullback $f^*(\gamma_k^n)$ under a map $f : B \to G_k(R^n)$, for some n. Homotopic maps $B \to G_k(R^n)$ induce isomorphic bundles; conversely, if $f^*\gamma_k^n \approx g^*\gamma_k^n$, then f is homotopic to g (as maps to $G_k(R^{2n})$). In symbols, the pullback of γ_k over $G_k(R^\infty)$ establishes a bijection:

$$Vect_k(B) \leftrightarrow [B, G_k(R^\infty)].$$

(Isomorphism classes of k-vector bundles over B on the left, homotopy classes of maps on the right.)

The same is true for complex vector bundles:

$$Vect_k^{\mathbb{C}}(B) \leftrightarrow [B, G_k(\mathbb{C}^\infty)].$$

In particular for line bundles:

$$Vect_1(B) \leftrightarrow [B, RP(\infty))] = [B, K(\mathbb{Z}_2, 1)].$$

$$Vect_1^{\mathbb{C}}(B) \leftrightarrow [B, \mathbb{C}P(\infty)] = [B, K(\mathbb{Z}, 2)].$$