ALGEBRAIC TOPOLOGY II-LECTURES, SPRING 2024

Lecture 6.

Definition of $\pi_n(X)$, $\pi_n(X,A)$. Group operations, commutativity for $n \geq 2$.

Meaning of $[f]_{(X,A)} = 0$: 'compression principle'

Homotopy exact sequence: proof of exactness.

Lecture 7. Preparation for Whitehead's Theorem.

Homotopy extension property for CW pairs (X, A) [Hatcher, Hilton]

Compression Lemma: $(X, A), (Y, B)$ connected CW pairs, $B \neq \emptyset$.. Assume $\pi_n(Y, B, y_0) = 0$ $(\forall y_0 \in B)$, for each n s.t. $X \setminus A$ has an n-cell. Then any map $(X, A) \to (Y, B)$ is homotopic (Rel. A) to a map $X \to B$.

Related facts (ref: Whitehead ch.2)

1. $\pi_q(Y, B) = 0, q = 1, \ldots, n \Leftrightarrow$ any map of a CW complex K into Y is htopic to a map mapping K^n to B.
2. $\pi_q(Y) = 0, q = 1, \ldots, n \Leftrightarrow$ any map of a CW complex K into Y is htopic to a map mapping K^n to the basepoint y_0.

The mapping cylinder M_f of $f : X \to Y$: deformationn retracts to Y, contains X as closed subspace $X \times 0$. Any map $f : X \to Y$ factors as inclusion to M_f, followed by homotopy equivalence $M_f \to Y$. (ref: Hatcher)

Lecture 8.

Whitehead's theorem: X, Y path-connected CW complexes, $f : X \to Y$. If $f_* : \pi_n(X) \to \pi_n(Y)$ is iso for all n, then f is a homotopy equivalence. If f is the inclusion map of a subcomplex $X \subset Y$, then X is a deformation retract of Y.

Proof. (ref: Hatcher) (i) For the subcomplex case, use the exact homotopy sequence to conclude the relative htopy groups all vanish, hence by the compression principle applied to the identity map $(X, Y) \to (X, Y)$, the identity can be deformed (rel. Y) to a map $X \to Y$, a deformation retraction. (ii) In the general case, use the mapping cylinder (a CW complex *if f is a cellular map*) to reduce to the inclusion case.

Remark: There is no 'relative version' of this theorem.

Example. $S^m \times RP^n, S^n \times RP^n$ $(m \neq n)$ have isomorphic homotopy groups, but not always isomorphic homology groups, hence are not htopy equivalent manifolds (use Künneth's formula or orientability.)

Example. $\pi_n(X \times Y) = \pi_n(X) \oplus \pi_n(Y)$ (ref: Hilton)

Example. $\pi_n(X \vee Y)$ (ref: Hilton)

Lecture 9.

Cellular approximation theorem. K, P CW complexes, $L \subset K$ subcomplex, $f_0 : K \to P$ cellular on $L \Rightarrow f_0 \simeq f_1$ (rel. L), where $f_1 : K \to P$ is cellular.

Cor.$\pi_m(S^q) = 0$ if $m < q$.

Cor. If Y is a CW complex with one vertex and no other cells of dimension $< q$, and X is a CW complex of dimension $< q$, any cits map $X \to Y$ is homotopic to a constant.

Cor. If $f_0, f_1 : K \to P$ are homotopic maps of (finite) CW complexes, and if restricted to a subcomplex $L \subset K$ the homotopy is cellular, then the homotopy may be replaced by a cellular homotopy, without changing it in L. (A homotopy $K \times I \to P$ is celular if it maps each sekeleton K^n to P^{n+1}.)

Def.: weak homotopy equivalence. $f : X \to Y$ inducing *iso* $f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ for each n, x_0. (So Whitehead's theorem says that for CW complexes, they are homotopy equivalences.)

Proposition. (4.22 in Hatcher.). A weak homotopy equivalence $f : Y \to Z$ induces (via composition) bijections of homotopy sets (resp. pointed homotopy sets):

$$[X, Y] \to [X, Z] \qquad \langle X, Y\rangle \to \langle X, Z\rangle.$$

Def. X is *n-connected* if $\forall q \leq n$, $[S^q, X]$ has one element; equivalently, if $\pi_q(X) = 0$ for q in this range, or any $f : S^q \to X$ extends to D^{q+1}.

(X, Y) *n-connected* if $\pi_q(X, Y) = 0$, $q = 1, \dots n$.

$f : X \to Y$ n-connected if M_f (mapping cylinder) is; equivalently, if induced maps on π_q are *iso* in this range.

Lemma:(Fomenko-Fuchs p.51) (X, A) CW pair, A contractible $\Rightarrow$ the quotient projection $p : X \to X/A$ is a homotopy equivalence.

A more precise version: (Prop. 4.28 in [Hatcher]) If (X, A) (CW pair) is r-connected (see below for def.) with A s-connected, then $\pi_i(X, A) \to \pi_i(X/A)$ is *iso* for $i \leq r + s$, *epi* for $i = r + s + 1$.

(*Challenge:* give a direct proof of this, without appealing to the 'homotopy excision theorem in [Hatcher].)

Theorem. (F-F 1.5.9) X n-connected (finite) CW complex, $n \geq 0$ (in particular path connected.) Then X is htopy equivalent to a CW complex with only one vertex and no cells of dimensions 1,2,...,n.

Cor. If X is n-dimensional and Y is n-connected (CW complexes), then $[X, Y]$ and $\langle, X, Y\rangle$ have only one element. (Use the theorem and cellular approximation.)

Relative version. Def: a pair (X, A) is n-connected if given any relative CW complex (Y, B) with $dim(Y, B) \leq n$, any $f : (Y, B) \to (X, A)$ is compressible into A (that is, homotopic rel. B to a map taking values in A.)

Cor. An n-connected CW pair (X, A) is htopy eq. to a CW pair (X', A') s.t. A' contains the n-skeleton $(X')^n$. (See also [Hatcher, cor. 4.16])

Lecture 10. Cone and suspension constructions (of spaces and maps.)

Unlinked submanifolds of euclidean space. Definition of 'linking number'.

Geometric proof of Freudenthal suspension theorem. (See [F-F, 10.1])

Invariance of degree under suspension/Proof that $\pi_n(S^n) \approx \mathbb{Z}$, with isomorphism given by the degree.

$\pi_{k+n}(S^n)$ stabilizes starting from $n = k + 2$; stable homotopy π_k^s of spheres.

Other facts: $\pi_q(S^3) \approx \pi_q(S^2)$, $q \geq 3$ (easy consequence of the homotopy sequence for the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$). Serre's great theorem: $\pi_q(S^n)$ is finite, except in the cases $q = n$ (infinite cyclic) and $\pi_{4n-1}(S^{2n})$ (direct sum of infinite cyclic and finite abelian.)

The Whitehead product $\alpha \in \pi_m(X), \beta \in \pi_n(X) \mapsto [\alpha, \beta] \in \pi_{m+n-1}(X)$. The kernel of the suspension map $\pi_{2n-1}(S^n) \to \pi_{2n}(S^{n+1})$ is cyclic, generated by the Whitehead square $[s_n, s_n]$; here s_n is a generator of $\pi_n(S^n)$. (See [F-F, 10.5].) It follows that $\pi_4(S^3) = \mathbb{Z}_2$, and hence $\pi_1^s = \mathbb{Z}_2$.

Lecture 11.

Basepoints and homotopy. $\pi_1(X, x_0)$ acts on the based homotopy set $\langle Z, X\rangle$ (say on the left), denote the action by $\tau_\alpha, \alpha \in \pi_1$.

Prop. (ref: Hatcher) (Z, z_0) CW pair, X path-connected. Then the natural map $\langle Z, X\rangle \to [Z, X]$ induces a bijection of the orbit set $\langle Z, X\rangle/\pi_1(X, x_0)$ onto $[Z, X]$. In particular, bijection between $[S^n, X]$ (free homotopy classes) and the orbit space.

Remark. In the case $n = 1$, the action is given by conjugation: $\tau_\alpha(\beta) = \alpha\beta\alpha^{-1}$.

Def. A space X is n-simple (or 'abelian') if the action of π_1 on π_n is trivial.

Action of $\pi_1(A)$ on relative homotopy groups of a pair (X, A) (Prop: the action is by hom). Effect of the action on the homotopy exact sequence (ref: Whitehead.)

The Hurewicz map h_n from homotopy to homology (ref: Whitehead IV.3, Hatcher p. 369-373.) Proof that the map is a homomorphism; the 'homotopy-homology ladder'.

The elements $\tau_\alpha(\beta)$ $(\alpha \in \pi_1(A), \beta \in \pi_n(X, A))$ or $\pi_n(X)$ or $\pi_n(A)$ are in the kernel of h_n; groups $\pi'_n(X), \pi'_n(X, A)$. Prop: if $n = 2$ (when $\pi_2(X, A)$ is not nec. abelian) the action of π_1 satisfies: $\tau_\xi(\alpha) = \beta\alpha\beta^{-1}$ $(\alpha, \beta \in \pi_2(X, A)$, $\xi = \partial_*(\beta) \in \pi_1(A)$.)

Lecture 12.

Effect of cell addition (Hatcher Ex. 4.29, FF 11.1–11.3, Whitehead 5.1)

Postnikov Towers (Hatcher Ex 4.17)

Eilenberg-MacLane spaces $K(G, n)$: existence, uniqueness (Hatcher p.365/366)

Examples: $K(\mathbb{Z}^r, 1)$, $K(G, 1)$ with G f.g. abelian, $K(\mathbb{Z}, 2)$

Connections with Moore spaces: statement of the Thom-Dold theorem (see Hatcher, bottom of p.365)

Lecture 13.

Proof of Hurewicz theorem and corollaries

Fibre bundles: homotopy lifting property, homotopy exact sequence.

Classical examples: fibrations of spheres by spheres, Hopf fibration, $CP(\infty)$ is $K(\mathbb{Z}, 2)$.

Lecture 14.

Vector bundles: definition
Examples: canonical line bundles γ_1^n over P^n and CP^n, its orthogonal complement, tangent bundle, normal bundle of a submanifold,

Triviality criterion in terms if l.i. sections; proof that γ_1^n over P^n is not trivial.

Constructions: pullback bundle, product bundle, Whitney sum, $Hom(E, F)$, vector bundles with fiberwise inner products.

Isomorphism $TP^n \approx Hom(\gamma_1^n, (\gamma_1^n)^\perp)$

Isomorphism $TP^n \oplus \epsilon_1 \approx \bigoplus_{n+1} \gamma_1^n$; stable equivalence, K-theory interpretation.

Lecture 15.

G-principal bundles: definition (free fiber-preserving right G-action, transitive on fibers, with compatible local trivializations.)

Free G-actions on a manifold E and $p : E \to E/G$ as a principal bundle: criterion given by local sections. (In particular if G is compact.) Example: quotient of a Lie group G by a compact subgroup H gives a principal H-bundle.

Example: Lie groups are parallelizable.

Lecture 16. Classification of vector bundles.

Definition of bundle morphisms and isomorphism.

The canonical k-vector space bundle γ_k^n over $G_k(R^n)$; γ_k over $Gr_k(R^\infty)$.

Theorem 1: Given a k-vector bundle ξ over B, existence of $f : B \to Gr_k(R^n)$ (for some n) such that $f^*\gamma_k^n \approx \xi$ (proof when B is compact.)

Theorem 2: homotopy of two classifying maps $B \to Gr_k(R^n)$ implies isomorphism of pullback bundles.

Gauss maps of k-vector bundles to R^m and isomorphism to a pullback of γ_k^m.

The even-odd trick. Theorem 3: isomorphism of the pullback bundles implies homotopy of the classifying maps to $G_k(R^n)$ (where the homotopy takes place in $G_k(R^{2n})$.

Summary: the classification theorem.