MAYER -VIETORIS SEQUENCES AND THE PROOF OF POINCARÉ DUALITY IN DE RHAM COHOMOLOGY

Notation: Let $A \subset B$ be open sets in M. Given $\omega \in \Omega^p(B)$, denote by $\omega_A^B \in \Omega^p(A)$ the restriction of ω from B to A. If $\omega \in \Omega^p(A)$, denote by $\omega^0_{A,B} \in \Omega^p(B)$ the extension (from A to B) of ω by zero. This is a smooth form in B if $spt(\omega) \subset A$ (that is, is a closed subset of the open set A.) To see this, note it is certainly smooth in a neighborhood of any $x \in A$, since A is open. And also smooth in a neighborhood of $x \in B \setminus A = B \cap A^c$, since such an x has a neighborhood not intersecting $spt(\omega) \subset A$ (closed in A).

Source: Based on [Bott-Tu, pp.22–27 and 42–46], with details added.

1. Mayer-Vietoris for de Rham cohomology.

Let $U, V \subset M$ be open sets with $M = U \cup V$. Let $f_U + f_V \equiv 1$ be a subordinate partition of unity, $spt(f_U) \subset U, spt(f_V) \subset V$ (closed subsets of U resp. V, but in general not compact.)

We have a short exact sequence:

$$0 \to \Omega^p(M) \to \Omega^p(U) \oplus \Omega^p(V) \to \Omega^p(U \cap V) \to 0,$$

with maps given by:

$$\omega \in \Omega^p(M) \mapsto (\omega_U^M, \omega_V^M), \quad (\omega, \eta) \in \Omega^p(U) \oplus \Omega^p(V) \mapsto \eta^V_{U\cap V} - \omega^U_{U\cap V}.$$

Exactness at the left and middle spaces is clear. To verify surjectivity of the second map, consider, given $\omega \in \Omega^p(U \cap V)$, the p-forms:

$$\omega_U = (f_V\omega)^0_{U\cap V,U} \in \Omega^p(U), \quad \omega_V = (f_U\omega)^0_{U\cap V,V} \in \Omega^p(U).$$

Subtle point: $spt(f_V\omega) \subset spt(f_V) \cap spt(\omega)$ is a closed subset of V , so its zero extension ω_U from $U \cap V$ to U is a smooth p-form in U (and similarly ω_V is a smooth form in V). It is easy to see that, under the second map:

$$(-\omega_U, \omega_V) \mapsto (\omega_V)^V_{U\cap V} + (\omega_U)^U_{U\cap V} = f_U\omega + f_V\omega = \omega.$$

Thus by general homological algebra we get a long exact sequence in cohomology:

$$\ldots \to H^q(M) \xrightarrow{(j_U^*, j_V^*)} H^q(U) \oplus H^q(V) \xrightarrow{k_V^* - k_U^*} H^q(U\cap V) \xrightarrow{d^*} H^{q+1}(M) \to \ldots$$

Here the first two arrows (on the level of forms) are restriction maps, the pullbacks of inclusions:

$$j_U : U \to M, \quad j_V : V \to M, \quad k_V : U \cap V \to V, \quad k_U : U \cap V \to U.$$

The map d^* is defined as follows: let $\omega \in \Omega^p(U \cap V)$, $d\omega = 0$. Then $0{=}d(f_U\omega + f_V\omega) = df_U \wedge \omega + df_V \wedge \omega$ in $U \cap V$, so in $U \cap V$:

$$df_V \wedge \omega = -df_U \wedge \omega,$$

a $(p+1)$-form with closed support in the open set $U \cap V$. This form is clearly well-defined and smooth in $U \cup V = M$ (by the first expression in U, by the second expression in V). So we define:

$$d^*[\omega]_{U\cap V} = [df_V \wedge \omega]_M \in H^{p+1}(M)$$

2.Mayer-Vietoris for compactly supported de Rham cohomology.

Again we let $U, V \subset M$ be open sets with $M = U \cup V$. For forms with compact support, we have a short exact sequence in the same order as that for homology, with maps given by zero extensions:

$$0 \to \Omega_c^p(U \cap V) \xrightarrow{\epsilon^0_{U\cap V,U} \oplus \epsilon^0_{U\cap V,V}} \Omega_c^p(U) \oplus \Omega_c^p(V) \xrightarrow{-\epsilon^0_{U,M} + \epsilon^0_{V,M}} \Omega_c^p(M) \to 0.$$

(For open sets $A \subset B$ in M, we let $\epsilon^0_{A,B} : \Omega_c^p(A) \to \Omega_c^p(B)$ denote the zero-extension operator of compactly supported forms.) Exactness at the first and second steps is easy to see. To see the second map is surjective, let $\omega \in \Omega_c^p(M)$ be closed, $d\omega = 0$. With $f_U + f_V \equiv 1$ a partition of unity as above, note $f_U\omega \in \Omega_c^p(U), f_V\omega \in \Omega_c^p(V)$ both have compact support (since, for instance $spt(f_U\omega) \subset spt(f_U) \cap spt(\omega)$ is closed, contained the intersection of a set closed in U with one compact in M, hence a compact subset of U.) Then on M:

$$-\epsilon^0_{U,M}(-f_U\omega) + \epsilon^0_{V,M}(f_V\omega) = \omega,$$

so under the second map the pair $(-f_U\omega, f_V\omega)$ maps to ω.

Again, homological algebra yields a long exact Mayer-Vietoris sequence in compactly supported cohomology, in the same order as that of homology:

$$\ldots \to H_c^p(U \cap V) \to H_c^p(U) \oplus H_c^p(V) \to H_c^p(M) \xrightarrow{d_*} H_c^{p+1}(U \cap V) \to \ldots,$$

where the first two arrows are induced by zero-extension operators (the second one a signed addition, signs + for V, - for U.) The operator d_* is defined as follows: let $\omega \in \Omega_c^p(M)$ be closed, $d\omega = 0$. Then $df_V \wedge \omega = -df_U \wedge \omega$, showing that the form $df_V \wedge \omega \in \Omega_c^{p+1}(V)$ is in fact in $\Omega_c^{p+1}(U \cap V)$. So we let:

$$d_*[\omega]_M = [df_V \wedge \omega]_{U\cap V}.$$

3. A commutative pairing between the two Mayer-Vietoris sequences.

Write the two long M-V exact sequences just discussed, the second one in reverse order:

$$\ldots \to H^q(M) \xrightarrow{(j_U^*, j_V^*)} H^q(U) \oplus H^q(V) \xrightarrow{k_V^* - k_U^*} H^q(U \cap V) \xrightarrow{d^*} H^{q+1}(M) \to \ldots$$

$$\ldots \leftarrow H_c^{n-q}(M) \xleftarrow{-\epsilon^0_{U,M} + \epsilon^0_{V,M}} H_c^{n-q}(U) \oplus H_c^{n-q}(V) \xleftarrow{\epsilon^0_{U\cap V,U} \oplus \epsilon^0_{U\cap V,V}} H_c^{n-q}(U \cap V) \xleftarrow{d_*} H_c^{n-q-1}(M) \leftarrow \ldots$$

There are vertical pairings between cohomology spaces of the same spaces, given on the level of closed forms by integration of the wedge product (which has compact support) over the corresponding space. (In the case of the direct sums, the pairing is the integral over V minus the integral over U.)

Main Lemma. The pairings between corresponding spaces commute up to sign.

We make this statement more precise for each of the three squares involved in each section of the diagram, from left to right.

(a) Square I. Let $\mu \in \Omega^q(M)$, $\xi \in \Omega_c^{n-q}(U)$, $\eta \in \Omega_c^{n-q}(V)$. Commutativity of the pairings in square I means (recalling the notations defined at the very beginning of this note):

$$\int_V \mu_V^M \wedge \eta - \int_U \mu_U^M \wedge \xi = \int_M \mu \wedge (\epsilon^0_{V,M}(\eta) - \epsilon^0_{U,M}(\xi)).$$

This is clear, since, for example, $spt(\mu \wedge \epsilon^0_{V,M}(\eta))$ is a compact subset of V, and on V:

$$\mu_V^M \wedge \eta = \mu \wedge \eta = \mu \wedge \epsilon^0_{V,M}(\eta).$$

(b) Square II. Let $\alpha \in \Omega^q(U), \beta \in \Omega^q(V), \lambda \in \Omega_c^{n-q}(U \cap V)$ be closed forms. Commutativity of the pairings in square II means:

$$\int_V \beta \wedge \lambda - \int_U \alpha \wedge \lambda = \int_{U\cap V} (\beta^V_{U\cap V} - \alpha^U_{U\cap V}) \wedge \lambda.$$

This is also clear since, for example, $spt(\beta \wedge \lambda)$ is a compact subset of $U \cap V$, and on $U \cap V$:

$$\beta^V_{U\cap V} \wedge \lambda = \beta \wedge \lambda.$$

(c) Square III. Let $\omega \in \Omega^q(U \cap V)$, $\tau \in \Omega_c^{n-q-1}(M)$ be closed forms. Commutativity up to sign of square III means, on the level of cohomology classes:

$$\int_M d^*[\omega] \wedge [\tau] = \pm \int_{U\cap V} [\omega] \wedge d_*[\tau].$$

Recall that, using the fact $df_V \wedge \omega$ is defined not only on U, but in fact on M, we defined:

$$d^*[\omega]_{U\cap V} = [df_V \wedge \omega]_M.$$

And we also defined, using the fact $df_V \wedge \tau^M_{U\cap V}$ is an $(n-q)-$form of compact support in $U \cap V$:

$$d_*[\tau]_M = [df_V \wedge \tau^M_{U\cap V}]_{U\cap V}.$$

So we want to compare:

$$\int_M (df_V \wedge \omega) \wedge \tau \text{ and } \int_{U\cap V} \omega \wedge (df_V \wedge \tau^M_{U\cap V}).$$

Since $spt[df_V \wedge \omega) \wedge \tau] \subset U \cap V$, these two integrals clearly differ only by a sign.

4. Statement and proof of Poincare' duality.

Duality Theorem. Let M^n be an orientable manifold admitting a finite good cover (for example, a compact orientable n-manifold.) Then the pairing of de Rham cohomology spaces:

$$H^q(M) \otimes H^{n-q}_c(M) \to \mathbb{R}$$

(defined by integration of the wedge product over M) is nondegenerate. Since both vector spaces are finite dimensional (see [Bott-Tu, p.43]), this establishes an isomorphism:

$$H^q(M) \approx (H^{n-q}_c(M))^*.$$

Proof. (outline). First note that for $M = \mathbb{R}^n$, it follows from the Poincare' lemmas that the only nontrivial case is the pairing:

$$H^0(\mathbb{R}^n) \otimes H^n_c(\mathbb{R}^n) \to \mathbb{R}, \quad f \otimes gdvol_n \mapsto \int_{\mathbb{R}^n} fgdvol_n,$$

where $f \in \mathbb{R}$ is a constant function and g is a smooth function with compact support, so $gdvol_n \in \Omega^n_c(\mathbb{R}^n)$. This pairing is clearly nondegenerate.

The Main Lemma (combined with the Five Lemma) implies that if the integration pairing is nondegenerate for open sets U, V and $U \cap V$, it is also nondegenerate for $U \cup V$. And the case of $\mathbb{R}^n$ shows it is true for the intersection of any two sets in a good cover (since by definition of ‘good cover’ this intersection is diffeomorphic to $\mathbb{R}^n$). This implies the result, by induction on the number of sets in a good cover.