LECTURE NOTES ON OBSTRUCTION THEORY [1]

The goal is to classify homotopy classes of maps $K \to Y$ using the cohomology $H^*(K;G)$ for a suitable abelian group G depending on Y. K is taken to be a finite cell complex. The strategy is stepwise extension of maps to Y defined on the skeleta K^q of K ($q \geq 0$.)

The classical development has two main steps. In *Step 1* the obstruction cocycle $c(f) \in C^{q+1}(K;\pi_q)$ and difference cochain $d(f_0, f_1) \in C^q(K;\pi_q)$ are defined, where $f_0, f_1 : K^q \to Y$ and $\pi_q = \pi_q(Y)$. In this step Y is assumed to be path-connected and 'q-simple', that is, the action of $\pi_1(Y)$ on $\pi_q(Y)$ is trivial. Thus the homotopy group $\pi_q(Y)$ can be unambiguously identified with the set of homotopy classes $[S^q, Y]$, independent of the choice of basepoint. This assumption on Y is not unduly restrictive; for instance Lie groups, or quotients G/H of Lie groups by closed subgroups are q simple for all $q \geq 1$. Of course simply-connected Y are q-simple for all q. We assume $q \geq 1$ throughout, noting that 1-simple means $\pi_1(Y)$ is abelian (hence may be used as a coefficient group.)

Definition of the obstruction cochain. Imagine building a map from K to Y. We can define it arbitrarily on K^0, and since Y is path-connected, easily extend it to K^1. Suppose $f : K^q \to Y$ is given; next we want to extend it to each $(q+1)$-cell σ. Let $h_\sigma : D^{q+1} \to K^q$ be its characteristic map, restricting on the boundary S^q to the attachment map $\phi_\sigma : S^q \to K^q$. The composition $f \circ \phi_\sigma$ defines an element of $\pi_q = \pi_q(Y)$, and f extends to σ iff this element is zero. Thus we have defined a homomorphism from $C_{q+1}(K)$ (cellular chain complex) to π_q, that is, a cochain:

$$c(f) \in C^{q+1}(K;\pi_q), \quad c(f)(\sigma) = [f \circ \phi_\sigma] \in \pi_q(Y),$$

the *obstruction cochain* of f. By definition, f extends to K^{q+1} iff $c(f) = 0$.

Two properties of $c(f)$ are immediate. First, naturality under mappings: Given $h : K \to K'$ and $f' : (K')^q \to Y$, if we define $f = f' \circ h : K \to Y$, it is clear that:

$$c(f) = h^* c(f'),$$

for the usual induced map on cochains, $h^* : C^{q+1}(K';\pi_q) \to C^{q+1}(K;\pi_q)$. Also, if $f_0, f_1 : K^q \to Y$ are homotopic, we have $c(f_0) = c(f_1)$ (since f_0, f_1 are homotopic on the boundary of each (q+1)-cell, hence their compositions with the attachment maps define the same element of π_q.) The following property is less easy:

Theorem 1: $\delta c(f) = 0$: $c(f)$ is a cocycle.

Proof. First consider the special case where K is assumed to be $(q-1)$-connected. Then if $f : K^q \to Y$ is given, we claim $c(f)$ is a *coboundary.* To see this, consider the isomorphism $\psi : \pi_q(K^q) \to Z_q(Y)$ defined as follows: since K^q is also $(q-1)$-connected (by cellular approximation), from Hurewicz we have

[1] Algebraic Topology II, April 2024, U.T.K. Sources given at the end of the note.

an isomorphism $\psi : \pi_q(K^q) \to H^q(K^q)$. But $H^q(K^q) = Z^q(K^q)$, since there are no $(q+1)$-chains in K^q (using cellular homology.) Thus in this case $c(f)$ admits the alternative description: consider the composition of the maps:

$$C_{q+1}(K) \xrightarrow{\partial} Z_q(K) \xrightarrow{\psi^{-1}} \pi_q(K^q) \xrightarrow{f_*} \pi_q(Y).$$

Since $C_q(K)$ is free abelian, $Z_q(K) = ker(\partial)$ is a direct summand, so $f_*\psi^{-1}$ extends to a homomorphism $h : C_q(K) \to \pi_q(Y)$, that is, $h \in C^q(K; \pi_q)$. It is easy to see that $c(f)(\sigma) = h(\partial\sigma)$ for any $(q+1)$-cell σ, so $c(f) = \delta h$.

Turning now to the general case: let ζ be any $(q+2)$-cell in K, and consider the subcomplex $K' \subset K$ consisting of ζ and all its faces. Let $c' = c(f)_{|K'}$ Then $\delta c'$ has the same values on ζ as $\delta c(f)$, and in fact $c' = c(f')$, where $f' = f_{|K'}$. Since K' is $(q-1)$-connected (indeed contractible), we have that c' is a coboundary, in particular $(\delta c)'(\zeta) = 0$, so $\delta c(f)(\zeta) = 0$. □

Definition of the difference cochain. Suppose now $f_0, f_1 : K^q \to Y$ given, and homotopic on K^{q-1}; let $k : K^{q-1} \times I \to Y$ be the homotopy. Consider the product cell complex $K^I = K \times I$. Its q-skeleton is:

$$(K^I)^q = (K^{q-1} \times I) \cup (K^q \times \{0, 1\}),$$

and we see that the maps f_0, f_1 and k combine to define $F : (K^I)^q \to Y$. It is natural to consider the obstruction to extending F to $(K^I)^{q+1}$, namely the cocycle $c(F) \in C^{q+1}(K^I; \pi_q)$. Now, the assignment $\sigma \mapsto \sigma \times I$ is a bijective correspondence between q-cells of K and $(q+1)$-cells of K^I. Thus we may consider the map that to each σ assigns $c(F)(\sigma \times I) \in \pi_q$. This defines a cochain:

$$d(f_0, k, f_1) \in C^q(K; \pi_q), \quad d(f_0, k, f_1)(\sigma) = (-1)^{q+1}c(F)(\sigma \times I), \quad \sigma \in C_q(K) \text{ a cell.}$$

the *deformation cochain.* In the special case $f_{0|K^{q-1}} = f_{1|K^{q-1}}$, with k the homotopy that doesn't move anything, we use the notation $d(f_0, f_1) \in C^q(K; \pi_q)$, and call it the *difference cochain.*

By definition, $d(f_0, k, f_1) = 0$ or $d(f_0, f_1) = 0$ mean that $f_0 \simeq f_1$ on K^q.

Two easy properties are: (i) invariance under mappings: if $h : K' \to K$ and given f_0, k, f_1, with f_0', k', f_1' defined by composition with h, then:

$$h^* d(f_0, k, f_1) = d(f_0', k', f_1');$$

and (ii) the *addition formula* (or 'stacking of homotopies'): given $f_0, f_1, f_2 : K^q \to Y$, if $f_0 \simeq f_1$ and $f_1 \simeq f_2$ on K^{q-1} (via homotopies k, k' resp.)–from which it follows that $f_0 \simeq f_1$ on K^{q-1} (via the 'concatenated homotopy' k'')–the corresponding deformation cochains satisfy:

$$d(f_0, k'', f_2) = d(f_0, k, f_1) + d(f_1, k', f_2).$$

The cochain $d(f_0, k, f_1)$ is not a cocycle in general; something more interesting happens:

Theorem 2, coboundary formula. $\delta d(f_0,k,f_1) = c(f_0) - c(f_1)$.

Proof. Let $\tau \in C_{q+1}(K)$ be a cell, set $d = d(f_0,k,f_1)$. Then:

$$(\delta d)(\tau) = d(\partial\tau) = (-1)^{q+1}c(F)((\partial\tau)\times I) = (-1)^{q+1}c(F)[\partial(\tau\times I) - (-1)^{q+1}(\tau\times\partial I)]$$

$$= -c(F)[(\tau\times 1) - (\tau\times 0)] = [c(f_0) - c(f_1](\tau),$$

where in the fourth equality we used the fact that $c(F)$ is a cocycle. □

The following lemma is quite useful.

Lemma 1. Let $f_0 : K^q \to Y$ and $d \in C^q(K;\pi_q)$ be given. Then we may find $f_1 : K^q \to Y$ such that $f_{1|K^{q-1}} = f_{0|K^{q-1}}$ and $d(f_0,f_1) = d$.

Before proving this, we recall the standard cell decomposition of $L = S^q$: one zero-cell (a point * on the equator), two closed q-cells E_1, E_0 (upper/lower hemisphere), and one (closed) $(q-1)$-cell the equator S^{q-1}. We *claim* that given $\alpha \in \pi_q(Y,y_0)$ and a map $f : (E_0,*) \to (Y,y_0)$, we may find an extension $g : (L,*) \to (Y,y_0)$ of f, which represents the homotopy class α.

To see this, let $g_0 : (L,*) \to (Y,y_0)$ be any map representing α. Since E_0 is contractible, $g_{0|E_0} \simeq f$ (say g_0 at $t=0$, f at $t=1$.) Since $E_0 \subset L$ is a subcomplex, the homotopy extends to all of L, yielding that g_0 (at $t=0$) is homotopic to a map $g : L \to Y$ (at $t=1$), which equals f on E_0. Since g (being homotopic to g_0 on L) also represents α, this proves the claim.

Proof of Lemma 1. The map f_0 defines a map $F_0 : (K^q\times 0)\cup(K^{q-1}\times I) \to Y$, $F_0(x,t) = f_0(x)$. We seek an extension F of F_0 to $(K\times I)^q$ such that the value of the obstruction cocycle $c(F) \in C^{q+1}(K\times I;\pi_q)$ on a cell $\tau\times I$, $\tau \in C_q(K)$, equals $(-1)^{q+1}d(\tau)$; for then $d(f_0,f_1) = d$, where $f_1 : K^q \to Y$ is the map defined by the extension F at $t=1$.

To find this extension cell by cell, let τ be a q-cell of K, with characteristic map $h_\tau : (D^q,S^{q-1}) \to (K^q,K^{q-1})$. We now think of another model for the cell decomposition of $S^q \approx \partial(D^q\times I)$, namely with q-cells:

$$E_0 = (D^q\times 0)\cup(S^{q-1}\times I), \quad E_1 = D^q\times 1.$$

Consider the composition of F_0 with the restriction of $h_\tau\times id$ to E_0:

$$E_0 \xrightarrow{h_\tau\times id} (K^q\times 0)\cup(K^{q-1}\times I) \xrightarrow{F_0} Y.$$

By the claim discussed just before the proof, we may find an extension F_τ of this composition to a map $E_0\cup E_1 \to Y$ representing the element $(-1)^{q+1}d(\tau)$ in $\pi_q(Y)$. It suffices then to define F on $\tau\times 1$ as $F_{\tau|E_1}$. This extends F_0 to F defined on $(K^q\times\{0,1\})\cup(K^{q-1}\times I) = (K\times I)^q$. □

Next, for $f' : K^q \to Y$ we interpret the meaning of the vanishing of the cohomology class $\bar{c}(f) \in H^{q+1}(K;\pi_q)$ of the obstruction cocycle $c(f')$ (the 'primary obstruction class' of f'). Here $f = f'_{|K^{q-1}}$

Theorem 3. Let $f : K^{q-1} \to Y$; assume f extends to K^q. Then the set $\{c(f')\} \subset C^{n+1}(K;\pi_q)$ of obstruction cocycles of all such extensions spans a single cohomology class $\bar{c}(f) \in H^{q+1}(K;\pi_q)$; and f admits an extension to K^{q+1} iff $\bar{c}(f) = 0$.

Proof. (i) Let f_0, f_1 be two extensions of f to K^q. Then $d(f_0, f_1) \in C^q(K;\pi_q)$ is defined, and the coboundary formula show $c(f_0), c(f_1)$ are cohomologous.

(ii) Let f_0 be an extension of f to K^q, $c \in C^{q+1}(K;\pi_q)$ a cocycle cohomologous to $c(f_0)$; thus there exists a q-cochain d so that $c(f_0) - c = \delta d$. By Lemma 1, we may find an extension f_1 of f to a map $K^q \to Y$ so that $d(f_0, f_1) = d$. Then the coboundary formula implies $c(f_0) = c$.

(iii) If f extends to K^{q+1} and f' is an extension, then $f_0 = f'_{|K^q}$ extends to K^{q+1}, so $c(f_0) = 0$, so $\bar{c}(f) = 0$.

(iv) Conversely, assuming $\bar{c}(f) = 0$, we have an extension f' of f to K^q such that $c(f') = 0$. Thus f' is extendible to K^{q+1}. □

Thus if $f : K^q \to Y$ is given, then $f_{|K^{(q-1)}}$ extends to K^{q+1} iff $c(f)$ is a coboundary. In terms of the stepwise extension process, this means: if we get to a map $f : K^q \to Y$ and find that $c(f) \neq 0$ (so that we can't extend f directly to K^{q+1}), if $c(f)$ is cohomologous to 0 it is possible to change f on the q-cells of K (leaving it alone on K^{q-1}) to obtain a new map on K^q, which *is* extendible to K^{q+1}. (A bit like rock climbing, of which Hassler Whitney was a practitioner.)

There is an analogous interpretation for the deformation cochains, giving information on the extension of homotopies (of maps defined on all of K).

Theorem 4. Let $f_0, f_1 : K \to Y$, and assume: $f_{0|K^{q-2}} \simeq f_{1|K^{q-2}}$ via $k : K^{q-2} \times I \to Y$. Assume k extends to a homotopy k' from f_0 to f_1, defined on K^{q-1}. Then the set $\{d(f_0, k', f_1)\}$ of deformation *cocycles* of all such extensions spans a single cohomology class in $Z^q(K;\pi_q)$, defining $\bar{d}(f_0, k, f_1) \in H^q(K;\pi_q)$. (Note they are cocycles by the coboundary formula, since the obstructions $c(f_0), c(f_1)$ vanish, given f_0, f_1 are assumed globally defined on K.) The homotopy k extends to a homotopy $f_{0|K^q} \simeq f_{1|K^q}$ iff $\bar{d}(f_0, k, f_1) = 0$.

Thus if k is a homotopy from the restriction of f_0 to K^{q-1} to the restriction of f_1 to K^{q-1}, $k_{|K^{q-2} \times I}$ extends to a homotopy on K^q from f_0 to f_1 iff $d(f_0, k, f_1)$ is a coboundary. (But we may need to modify the homotopy on the $(q-1)$-cells.) The corresponding result for the difference cochain is: if $f_0, f_1 : K \to Y$ coincide on K^{q-1}, then $f_{0|K^q} \simeq f_{1|K^q}$ (rel. K^{q-2}) iff the q-cocycle $d(f_0, f_1)$ is a coboundary.

Now recall that homotopies defined on a subcomplex (such as K^q) extend to homotopies defined on all of K. We obtain:

Theorem 5. Let $f_0, f_1 : K \to Y$, and assume $f_0 = f_1$ on K^{q-1} (so $d(f_0, f_1)$ is defined). Then there exists $f'_1 : K \to Y$ homotopic to f_0 on K (rel. K^{q-2}) and coinciding with f_1 on K^q iff the difference *cocycle* $d(f_0, f_1) \in Z^q(K, \pi_q)$ is a coboundary.

This concludes 'Step 1' of the obstruction theory program.

Moving to **Step 2**, we make stronger assumptions on Y: path-connected, simply-connected if $q > 1$, abelian π_1 if $q = 1$ and, most important, $(q-1)$-connected: all homotopy groups π_r are trivial for $r = 0, 1, \dots q-1$ ($q \geq 1$). We also assume $\pi_q = \pi_q(Y) \neq 0$.

This hypothesis implies that, defining $f : K^0 \to Y$ arbitrarily, we can always extend it to $f : K^q \to Y$. And if $f_1, f_2 : K^q \to Y$ are two such extensions, $c(f_1) - c(f_2) \in C^{q+1}(K; \pi_q)$ is a coboundary: namely, $f_{1|K^{q-1}} \simeq f_{2|K^{q-1}}$ (again by the hypothesis on Y), say via a homotopy k, implies the q-cochain $d(f_1, k, f_2)$ is defined, and its coboundary is $c(f_1) - c(f_2)$.

Definition: If $f : K^q \to Y$ is a (continuous) map, the obstruction class $\bar{c} = \bar{c}(f) \in H^{q+1}(K; \pi_q)$ is independent of f (as seen in the previous paragraph). We call this class $\bar{c}$ the *primary obstruction class* of the pair (K, Y). Its vanishing is necessary and sufficient for the extendability of maps from $K^q \to Y$ to $K^{q+1} \to Y$.

This class is natural under maps $h : K \to K'$, and in particular a topological invariant (independent of the cell decomposition of K.)

With the same hypotheses on Y, consider the problem: given maps $f_0, f_1 : K \to Y$, when are they homotopic? First, the assumptions on Y are easily seen to imply:

Lemma 2. Let $f_0, f_1 : K \to Y$. Then $f_{0|K^{q-1}} \simeq f_{1|K^{q-1}}$ via a homotopy k. If k, k' are two such homotopies, the difference of q-cochains: $d(f_0, k, f_1) - d(f_0, k', f_1)$ (which is a cocycle, in view of the coboundary formula) is in fact a coboundary.

Definition. The common cohomology class of the $d(f_0, k, f_1)$ (which, as just noted, is independent of k), denoted $\bar{d}(f_0, f_1) \in H^q(K, \pi_q)$, is the *primary difference class* of f_0, f_1. Its vanishing is necessary and sufficient for the existence of a homotopy from $f_{0|K^q}$ to $f_{1|K^q}$.

Analogous to Theorem 5, vanishing of $\bar{d}(f_0, f_1)$ is necessary and sufficient for the existence of $f_1' : K \to Y$ which coincides with f_1 on K^q and is homotopic to f_0 on K.

The difference class $\bar{d}(f_0, f_1)$ has the properties:

(i) Naturality: given $h : K \to K'$ and $f_0', f_1' : K' \to Y$, and defining $f_0, f_1 : K \to Y$ by composition with h, we have:

$$h^* \bar{d}(f_0', f_1') = \bar{d}(f_0, f_1).$$

(ii) Addition formula: given $f_0, f_1, f_2 : K \to Y$, their primary difference classes satisfy:

$$\bar{d}(f_0, f_2) = \bar{d}(f_0, f_1) + \bar{d}(f_1, f_2).$$

APPLICATIONS: classification theorems for homotopy classes. (Same hypotheses on Y as in Step 2.)

Lemma 3. Existence of maps with given primary difference class. Let $f_0 : K \to Y$. Assume $dim(K) \leq q+1$. Then for each $d \in H^q(K;\pi_q)$ there exists a map $f_1 : K \to Y$ such that $\bar{d}(f_0, f_1) = d$.

Proof. Let $d' \in Z^q(K;\pi_q)$ be a cocycle representing d. By Lemma 1, we know $f_{0|K^{q-1}}$ admits an extension $f_1 : K^q \to Y$ with $d(f_0, f_1) = d'$. But then the coboundary formula implies:

$$c(f_{0|K^q}) - c(f_1) = \delta(f_0, f_1) = \delta d' = 0.$$

Since f_0 is defined on all of K, $c(f_0) = 0$. Thus $c(f_1) = 0$ as well, and f_1 extends to $K^{q+1} = K$ (since we assume $dim(K) \leq q+1$.) □

Theorem 6: classification theorem. Let $dim(K) = q$, and fix some $f_0 : K \to Y$ (for example, a constant map). The assignment $[f] \mapsto \bar{d}(f, f_0)$ defines a bijection:

$$[K, Y] \to H^q(K, \pi_q).$$

Proof. (i) If $f \simeq f'$ on K, $\bar{d}(f, f_0) = \bar{d}(f', f_0)$; thus the map is well-defined.

(ii) If $\bar{d}(f, f_0) = \bar{d}(f', f_0)$, then by the addition formula $\bar{d}(f, f') = 0$. Since $K = K^q$, recalling the interpretation of this vanishing (given right after the definition of $\bar{d}$) we see that $f' \simeq f$ on K; thus the map is injective.

(iii) Given a cocycle $d \in Z^q(K;\pi_q)$, by Lemma 1 we may find $f : K \to Y$ extending $f_{0|K^{q-1}}$ such that $\bar{d}(f_0, f) = -[d]$ (cohomology class). Then the addition formula yields $\bar{d}(f, f_0) = d$. This shows the map is surjective. □.

In particular, consider the case where f_0 is a constant map, $f = c_{y_0}$ (and Y is a finite complex). Define the class $\bar{d}_Y = \bar{d}(c_{y_0}, id_Y) \in H^q(Y;\pi_q)$. (Say y_0 is a vertex of Y.) Since Y is connected, this cohomology class is independent of the choice of y_0. It is easy to see (by naturality) that if $f : K \to Y$ is any map, we have: $\bar{d}(c_{y_0}, f) = f^*\bar{d}_Y$. Thus we have:

Corollary 1: Hopf-Whitney theorem. Let $dim(K) = q$, and assume Y is a $(q-1)$-connected complex and $\pi_q = \pi_q(Y) \neq 0$. Then the map $f \mapsto f^*\bar{d}_Y$ defines a bijection:

$$\psi : [K, Y] \to H^q(K;\pi_q).$$

This generalizes the original Hopf theorem (where $Y = S^q$ and $\pi_q = \mathbb{Z}$.)

Remark 1. In the statement and proof of the Hopf-Whitney theorem, it is enough to consider homotopies that are fixed on K^{n-2}. In general, assuming Y is (q-1))-connected, denote by $\langle K, Y\rangle$ the set of equivalence classes of maps $K \to Y$, where 'equivalent' means 'homotopic by a homotopy fixing K^{q-1}'. Then the natural map $\langle K, Y\rangle \to [K, Y]$ is a bijection (see [Prasolov, p. 119].) We'll ignore this distinction in the notation, and just use $[K, Y]$, with 'homotopy constant on K^{q-2}' left implicit.

Remark 2/Definition. Let Y be a $(q-1)$-connected, path-connected simplicial complex, $q \geq 2$. Denoting by $h : \pi_q(Y) \to H_q(Y)$ the Hurewicz isomorphism, we have $h^{-1} \in Hom(H_q(Y), \pi_q(Y))$. And since $H_{q-1}(K) = 0$, by the universal coefficient theorem we have $H^q(Y;G) \approx Hom(H_q(Y);G)$, for any abelian coefficient group G. In particular, let $G = \pi_q(Y) = \pi_q$. We denote by $F_Y \in H^q(Y;G)$ the class corresponding to h^{-1} under this isomorphism, the *fundamental class* of Y.

Exercise. (i) Show that F_Y coincides with the class $d_Y \in H^q(Y;\pi_q)$ defined earlier (the primary difference class from a constant map to the identity).

(ii) Show that if Y is an oriented, $(q-1)$-connected manifold and $\pi_q(Y) = \mathbb{Z}$ (for example, $Y = S^q$), this definition coincides with the more common definition of 'fundamental class'.

Cohomology and maps to Eilenberg-McLane spaces $K(G,q)$.

It turns out that, starting from the Hopf-Whitney theorem, we may relax the dimension restriction on the domain complex K, at the cost of assuming the target space Y has non-vanishing homotopy groups in exactly one dimension; that is, by assuming Y is an Eilenberg-MacLane space $K(G,q)$, for the abelian group G. Recall $Y = K(G,q)$ is a path-connected CW complex (unique up to homotopy type, and usually infinite) with only one non-vanishing homotopy group, namely $\pi_q(Y) = G$. That is, we have the following fundamental theorem:

Theorem 7. Let $Y = K(G,q)$, K any finite simplicial complex. Denote by $F_Y \in H^q(Y;G)$ the fundamental class of Y. The map

$$\varphi : [K,Y] \to H^q(K;G), \quad \varphi([f]) = f^* F_Y$$

is a bijection.

Proof. (Following [Prasolov]). The idea is summarized in the following beautiful diagram: [2]

$$\begin{array}{ccc} \pi(K,Y) & \xrightarrow{i^\#} & \pi(K^q,Y) \\ \Big\downarrow{\scriptstyle\varphi} & & \Big\downarrow{\scriptstyle\psi} \\ H^q(K;G) & \xrightarrow{i^*} & H^q(K^q;G) \end{array}$$

Here ψ is the Hopf-Whitney map (a bijection), $i^\#$ and i^* are induced by the inclusion $K^q \hookrightarrow K$.

First, i^* *is injective:* the cocycles of dimension $\leq q$ in K are cocycles in K^q, and if such a cocycle (say z) is cohomologous to zero in K^q, say $z = \partial w$ where $w \in C^{q-1}(K^q;G)$, then in particular this is also true in $H^q(K;G)$.

Second, $i^\#$ *is injective:* if $f, g : K \to Y$ have homotopic restrictions to K^q, then they are homotopic as maps $K \to Y$, since the obstructons to extending homotopies to higher-dimensional skeleta lie in cocycles with values in $\pi_r(Y)$ with $r > q$, and those are assumed to vanish.

[2]We change notation here, writing $\pi(K,Y)$ for $[K,Y]$, since tikzcd doesn't like square brackets for some reason.

Thus, to complete the diagram with the map φ (simultaneously showing it is a bijection), it is enough to show that ψ maps the image of $i^{\#}$ to the image of i^* (and then necessarily bijectively.)

To prove this, we use the model of $Y = K(G, q)$ with (q-1)-skeleton a vertex y_0, and q-skeleton a wedge of q-spheres with common point y_0 (one for each generator of G.) Thus (by cellular approximation of maps) it is enough to consider maps $f : (K^q, K^{q-1}) \to (Y, y_0)$, up to homotopy constant on K^{q-2}. The homotopy class of such f is in $im(i^{\#})$ iff f extends to K^{q+1} (since the obstruction cocycles to further extension have image in trivial homotopy groups). In that case, the obstruction cocycle $c(f) \in C^{q+1}(K; G)$ vanishes. In view of the coboundary formula: $\delta d(f, c_{y_0}) = c(f) - c(c_{y_0})$, we have $\delta d(f, c_{y_0}) = 0$ (since of course $c(c_{y_0}) = 0$), thus $d(f, c_{y_0})$ is *a cocycle in* K. Recall that, by definition, $\psi([f])$ is the cohomology class of the cocycle $d(f, c_{y_0})$ in $H^q(K^q; G)$. Since we just showed this is also a cocycle in K, it follows that, indeed, $\psi([f]) \in im(i^*)$.

It remains to show that the map φ has the form claimed. Note that by construction the map φ is natural: if $h : K \to K'$ induces $h^{\#} : [K', Y] \to [K, Y]$ and $h^* : H^q(K'; G) \to H^q(K, G)$, the maps:

$$\varphi_{K'} : [K', Y] \to H^q(K'; G), \quad \varphi_K : [K, Y] \to H^q(K; G)$$

satisfy: $\varphi_K \circ h^{\#} = h^* \circ \varphi_{K'}$.

For the model of $Y = K(G, q)$ referred to above, it's easy to see that $\varphi_Y([id_Y]) = F_Y$. Thus, for $f : K \to Y$, we find:

$$\varphi_K([f]) = (\varphi_K \circ f^{\#})(id_Y) = (f^* \circ \varphi_Y)(id_Y) = f^*(F_Y),$$

as claimed.

Sources. Most of this follows closely [Steenrod 1950, 32–27.] There, however, the existence of sections of fiber bundles is considered, which necessitates introducing local coefficients. Here we deal with the simpler case of maps, and only with absolute extensions (i.e., not relative to fixed existing maps on a subcomplex L). This is the recommended source for most missing proofs. Some proofs were learned from [G. Whitehead 1978, V.5, V.6], and there is also material from [Prasolov.] Nothing in this note is remotely original of course, all of it having been understood by the mid-1940s. (Mainly codified by S. Eilenberg, although understood at the time also by N.Steenrod, H.Whitney and L.Pontrjagin, among others.)