Chapter IV, Paragraph 1: The Euclidean rotation group SO_n.

(A) The skew-field $\mathbb{H}$ of quaternions, $\mathbb{H} = \{y^1 + y^2 i + y^3 j + y^4 k; y^l \in \mathbb{R}\}$. (A review.)

(B) The group of unit-norm quaternions double-covers SO_3.

Consider the orthogonal decomposition $\mathbb{H} = \mathbb{R} \oplus J$, where J is the space of imaginary quaternions, a three-dimensional vector space over $\mathbb{R}$. Let G be the group of unit quaternions (topologically: the three-sphere.) Given $g \in G$, define $\psi_g(x) = gxg^{-1}$. This linear isometry of $\mathbb{H}$ preserves $\mathbb{R}$, and hence also J; since G is connected, ψ_g is an orientation-preserving isometry of $J \approx \mathbb{R}^3$, i.e. an element of SO_3. It is easy to see that the kernel of the homomorphism $v : g \mapsto \psi_g$ is ± 1 (since quaternions in the kernel commute with all imaginary quaternions, hence must be real, with norm 1.) Thus this map v induces a continuous bijection $w : P^3 \to SO_3$, hence a homeomorphism (since P^3 is compact.) In addition, $v : G = S^3 \to SO_3 \approx P^3$ is the 2-to-1 universal covering map.

The following fact is used later. Let $g = \cos\beta + k \sin\beta$. Then $\psi_g(i) = i \cos 2\beta + j \sin 2\beta$ (easy quaternion calculation). Thus, restricted to the circle $\cos\beta + k\sin\beta$, the map $\omega = \chi \circ v$ (where $\chi : SO_3 \to S^2$, $\chi(T) = T(i) \in S^2$ for $T \in SO_3$) covers the circle $\{ai + bk; a^2 + b^2 = 1\} \subset S^2 \subset J$ with degree 2.

Covering homotopy. We recall a differential topology fact (consequence of the local form of submersions):

Lemma 1.1 *Homotopies lift over submersions:* That is, if $\varphi : P^p \to Q^q$ is a smooth submersion ($p \geq q, P, Q$ compact manifolds) and we're given $f_0 : R \to P$ (R compact metric and f_0 cont., or R compact manifold and f smooth) and a homotopy $g_t : R \to Q$ with $g_0 = \varphi \circ f_0$, then there exists a lift $f_t : R \to P$ of g_t, meaning $\varphi \circ f_t = g_t$.

(For a proof, see [P, p.93-95].)

(C) The group SO_n is a smooth manifold of dimension $n(n-1)/2$. There exists a smooth submersion $SO_n \to S^{n-1}$, with fibers diffeomorphic to SO_{n-1}.

The fundamental group of SO_n is $\mathbb{Z}_2$ if $n \geq 3$. For $n = 3$ this was seen in (B) above. This implies the result for $n \geq 3$, since the homotopy exact sequence of the fibration $SO_{n-1} \hookrightarrow SO_n \to S^{n-1}$ is easily seen to imply $\pi_1(SO_n) \sim \pi_1(SO_{n-1})$ if $n \geq 4$. The maps $S^1 \to SO_n$ not homotopic to 0 may be described as follows. An orthogonal decomposition $E^n = E^2 \oplus E^{n-2}$ (where E^2 is any two-dimensional subspace) defines an embedding $SO_2 \hookrightarrow SO_n$.

Theorem 17. (i) Each $h : S^1 \to SO_n$ is homotopic (rel $h^{-1}(SO_2)$) to some $g : S^1 \to SO_2$.

(ii) A map $g : S^1 \to SO_2$ is homotopic to a constant in SO_n iff $deg(g)$ is even.

Proof. (i) Let $\chi : SO_n \to S^{n-1}$ be the usual subnersion ($\chi(T) = Te_1$.) Fix $a \in S^{n-1}$, so $\chi^{-1}(a) \sim SO_{n-1}$ (rotations in the hyperplane orthogonal to a.)

Claim. If $f_0 : M^r \to SO_n$ is smooth, $r \leq n-2$, there exists a homotopy f_t (rel. $f_0^{-1}(SO_{n-1})$ of f_0 in SO_n so that $f_1(M^n) \subset SO_{n-1}$.

This follows since $\chi \circ f_0(M^r)$ is nowhere dense in S^{n-1}, so there exists a homotopy g_t of $\chi \circ f_0$ in S^{n-1} (rel. the preimage of a) so that $g_1 : M^r \to S^{n-1}$ is the constant map to a. Lifting this to a homotopy $f_t : M^r \to SO_n$ yields the claim.

Now let $M^r = S^1$ and apply the claim iteratively to the inclusions $SO_{m-1} \hookrightarrow SO_m$, down to $m = 3$ to finish the proof of (i).

Remark: This proof is a nice illustration of how one can get around the 'cellular approximation theorem' (which quickly implies the conclusion (i)) in the smooth case.

To prove (ii), first observe that if $g : S^1 \to SO_2$ is $\simeq$ const. in SO_n, then this is so in SO_3 as well. For then g extends to $\hat{g} : D^2 \to SO_n$, and the claim in part (i), applied (iteratively) to $M^r = D^2$ shows $\hat{g}$ is homotopic (rel. $\hat{g}^{-1}(SO_3)$) to a map $h : D^2 \to SO_3$, which restricts to g on $S^1 = \partial D^2$.

So we need to show that $g : S^1 \to SO_2 \subset SO_3$ is $\simeq$ const in SO_3 iff $deg(g)$ is even. We use the double covering $v : S^3 \to SO_3$. Recall from (A) in this paragraph that $\Sigma^1 = v^{-1}(SO_2)$ is a circle in S^3, mapping to SO_2 with degree 2.

Assume $deg(g) = n = 2m$ is even. Let $h : S^1 \to \Sigma^1$ have degree m. Then $v \circ h : S^1 \to SO_2$ has degree $2m = n$, hence is homotopic (in SO_2, and *a fortiori* in SO_3) to g. Since $h \simeq$ const. in S^3, we have $v \circ h \simeq$ const in SO_3; hence also $g \simeq$ const. in SO_3.

Conversely, suppose $g : S^1 \to SO_2$ is $\simeq$ const. in SO_3, via $g_t : S^1 \to SO_3$ ($g_1 = g, g_0 \equiv q \in SO_3$). Pick $p \in S^3$ so that $v(p) = q$, and let $f_0 : S^1 \to S^3$ be the constant map to p. The homotopy g_t lifts (over the submersion $S^3 \to SO_3$) to $f_t : S^1 \to S^3$. So $v \circ f_1 = g_1 = g$, and f_1 maps S^1 to Σ^1, a circle. Since $deg(v) = 2$, it follows that $deg(g)$ is even. □

(D) The homotopy invariant β.

Let M^1 be a compact one-dimensional manifold. To each homotopy class of $h : M^1 \to SO_n$ we assign $\beta(h) \in \mathbb{Z}_2$, as follows: (i) if $n \geq 3$: if $M^1 = S^1$, $\beta(h) = 0$ if $h \simeq$ const., $\beta(h) = 1$ otherwise. If M^1 is not connected, we add the $\beta(h)$ mod 2 over the finitely many components. (ii) if $n = 2$: for $M^1 = S^1$, set $\beta(h)$ equal to the mod 2 reduction of $deg(h)$; then add mod 2 over components.

Given two maps $f, g : S^1 \to SO_n$, consider the product map $h(x) = f(x)g(x)$ (product in the group SO_n.) We *claim* the following:

$$\beta(h) = \beta(f)\beta(g) \text{ (product in } \mathbb{Z}_2.)$$

To see this, consider the map $\varphi : T^2 \to SO_n$, $\varphi(x, y) = f(x)g(y)$. Fix $a \in S^1$ such that $f(a) = g(a) = e \in SO_n$ (the identity) and define the maps from S^1 to SO_n:

$$f_a(x) = (x, a), \quad g_a(x) = (a, x), \quad \Delta(x) = (x, x).$$

Clearly, we have:

$$f = \varphi \circ f_a, \quad g = \varphi \circ g_a, \quad h = \varphi \circ \Delta.$$

Consider the cell structure in T^2 with one 0-cell (a), two 1-cells (the figure-eight space $S^1 \vee S^1$, two circles meeting at a) and one 2-cell. Clearly Δ is homotopic to the map $\hat{\Delta}$ from S^1 to the figure eight, which maps S^1 with degree 1 to each circle of the figure-eight. Thus we have:

$$\beta(h) = \beta(\varphi \circ \Delta) = \beta(\varphi \circ \hat{\Delta}) = \beta(\varphi \circ f_a) + \beta(\varphi \circ g_a) = \beta(f) + \beta(g).$$

(For the third equality, just recall β is the degree mod 2, for maps $S^1 \to SO_2$.)

$\square$

Chapter IV, Paragraph 2: Classification of maps $\Sigma^3 \to S^2$.

Theorem 18. Every map $S^n \to S^1$ for $n \geq 2$ is nullhomotopic.
Proof: follows directly from the invariance of higher homotopy groups under covering maps. (Or see [P, p.99-100 for a proof.])

Hopf mapping from the 3-sphere to the 2-sphere.

(A) We'll be led to consider normal framings in E^3 (with coordinates (y^1, y^2, y^3)) of the unit circle in E^2, parametrized as:

$$S^1 \subset E^2 \times 0 \subset E^3, \quad S^1 = \{x = (\cos\theta, \sin\theta, 0); \theta \in R\}.$$

The normal space at $x = (\cos\theta, \sin\theta, 0) \in S^1$ with coordinates (t_1, t_2) is:

$$N_x^2 = \{y_1 = (1+t^1)\cos\theta, y^2 = (1+t^1)\sin\theta, y^3 = t^3\},$$

and we have a background o.n. frame $U(x) = \{u_1(x), u_2(x)\} = \{x, e_3(x)\}$, with e_3 the third basis vector of E^3, (And [P] regards both vectors as 'parallel translated' to have basepoint x.)

We also have, for each $r \in \mathbb{Z}$, the rotated frame V_r, with normal vectors at x:

$$v_1(x) = (\cos r\theta)u_1(x) + (\sin r\theta)u_2(x), \quad v_2(x) = (-\sin r\theta)u_1(x) + (\cos r\theta)u_2(x).$$

It is easy to see that, for $c = (c_1, c_2) \in R^2$ small, the embedding of S^1 into E^3:

$$\eta_c(x) = x + c_1 v_1(x) + c_2 v_2(x)$$

wraps around the original S^1 r times; so it's not hard to show from the definition that the Hopf invariant of this frame is:

$$\gamma(S^1, V_r) = Lk(S^1, \eta_c(S^1)) = r.$$

Lemma 2.1: the Hopf map.
There exists a smooth submersion $\omega : \Sigma^3 \to S^2$ such that all preimages $\omega^{-1}(y), y \in S^2$, are circles, and for the Hopf invariant: $\gamma(\omega) = 1$.

Proof. Let $\mathbb{H}$ denote the skew-field of quaternions (with coordinates $y^1 + y^2 i + y^3 j + y^4 k$), $G \subset \mathbb{H}$ the group of unit norm quaternions (topologically the sphere Σ^3, isomorphic to SU_2) and $J \subset \mathbb{H}$ the imaginary quaternions, spanned over the real numbers by the unit quaternions i, j, k. For S^2 we take the. unit sphere in J.

Define $\omega : G \to S^2$ by $\omega(g) = gig^{-1}$. Then if $v : G \to SO_3$ is the double covering ($v(g) = T$ if $T(w) = gwg^{-1}, w \in J$) and $\chi : SO_3 \to S^2$ the submersion $\chi(T) = T(i)$, then $\omega = \chi \circ v$, so ω is a submersion. All preimages are homeomorphic, and since the preimage of i is the circle $\{\cos\theta + i\sin\theta\}$, they are all circles.

To find the frame $V(x), x \in S^1$, associated to the map ω, [P, p.101/102] uses quaternions in a computation. The final result is that, relative to the background normal frame $\{u_1(x), u_2(x)\}$ described above, we have at $x = \cos\theta + i\sin\theta \in \omega^{-1}(i)$:

$$v_1(x) = (\cos\theta)u_1(x) + (\sin\theta)u_2(x), \quad v_2(x) = (-\sin\theta)u_1(x) + (\cos\theta)u_2(x).$$

so $V(x) = V_1(x)$, which implies $\gamma(\omega) = \gamma(S^1, V_1) = 1$. $\square$.

Classification of mappings $\Sigma^3 \to S^2$.

The Hopf map ω induces the homomorphism of homotopy groups:

$$\omega_* : \pi_n(\Sigma^3) \to \pi_n(S^2), \quad [f] \mapsto [\omega \circ f], \quad f : S^n \to \Sigma^3.$$

Lemma 2.2: For $n \geq 3$, ω_* is an isomorphism.

Proof. Both parts of the proof appeal to Lemma 1.1 above (homotopies lift over submersions.)

(i) We show ω_* *is injective.* Let $f : S^n \to \Sigma^3$ be such that $\omega \circ f : S^n \to S^2$ is homotopic to a constant: there exists $g_t : S^n \to S^2$ so that $g_0 = \omega \circ f, g_1 \equiv c \in S^2$. Then we have a lift $f_t : S^n \to \Sigma^3$ with $f_0 = f, \omega \circ f_t = g_t$. Thus $f_1(S^n) \subset \omega^{-1}(c)$, a circle. This implies $f_1 \simeq$ const., and hence also $f \simeq$ const.

(ii) We show ω_* *is surjective.* Given $\beta \in \pi_n(S^2)$, we must find $f : S^n \to \Sigma^3$ such that $[\omega \circ f] = \beta$.
Let $S^n \subset E^{n+1}$ with E_+, E_- the upper/lower hemispheres, intersecting at $S^{n-1} = \{x_{n+1} = 0\}$, with north/south poles $p = e_{n+1}, q = -e_{n+1}$.

In the homotopy class β, we may choose a representative $g : S^n \to S^2$ mapping E_- to a point $c \in S^2$ (since there exists a map of S^n homotopic to the identity, taking E_- to q.) We now need to find $f : S^n \to \Sigma^3$ with $\omega \circ f = g$.

For $x \in S^{n-1}$, let Γ_x be the half-meridian (=half great circle) of S^n defined by p, x and q. Parametrize its intersection with E_+ so that $\Gamma_x \cap \{x_{n+1} = 1 - t\}$ corresponds to $(x, t), x \in S^{n-1}, t \in [0, 1]$. Defining $g_t : S^{n-1} \to S^2$ by $g_t(x) = g(x, t)$, we have a homotopy with $g_1 \equiv c \in S^2$, $g_0 \equiv b \in S^2$, where $b = g(p)$. Pick $a \in \omega^{-1}(b)$ and let $f_0 : S^{n-1} \to \Sigma^3$ be the constant map, $f_0(S^{n-1}) = \{a\}$. Thus $g_0 = \omega \circ f_0 \equiv b$.

We may lift the homotopy g_t over ω starting at f_0, obtaining $f_t : S^{n-1} \to \Sigma^3$ such that $\omega \circ f_t = g_t$. Using the same parameterization of $\Gamma_x \cap E_+$, we define $f : E_+ \to S^2$ via $f(x, t) = f_t(x)$. Thus $\omega \circ f = g$ on E_+. At $t = 1$, f maps S^{n-1} to $\omega^{-1}(c)$, a circle. Hence $f_{|S^{n-1}} \simeq$ const (say $d \in \omega^{-1}(c)$). This homotopy takes place in $\omega^{-1}(c)$, and we use it to extend f 'radially' to E_-, with values in $\omega^{-1}(c)$ and mapping q to d. Since $g \equiv c$ on E_-, we have $\omega \circ f = g$ on E_-. All told, we have found $f : S^n \to \Sigma^3$ with $\omega \circ f = g$. $\square$

Theorem 19. The homomorphism $\gamma : \Pi_2^1 \to \mathbb{Z}$ (Hopf invariant of a framed 1-manifold in E^3) is an isomorphism. Thus $f_0, f_1 : \Sigma^3 \to S^2$ are homotopic iff $\gamma(f_0) = \gamma(f_1)$.

Proof. (i) γ *is mono*. Let $g : \Sigma^3 \to S^2$ satisfy $\gamma(g) = 0$. We must show $g \simeq$ const. By Lemma 2, there exists $f : \Sigma^3 \to \Sigma^3$ such that $\omega \circ f \simeq g$; thus $\gamma(\omega \circ f) = 0$. But $\gamma(\omega \circ f) = \gamma(\omega) deg(f)$; thus $deg(f) = 0$. So $f \simeq$ const., hence $g \simeq$ const.

(ii) Given $n \in \mathbb{Z}$, let $f : \Sigma^3 \to \Sigma^3$ satisfy $deg(f) = n$. Then if $g = \omega \circ f$ we have $\gamma(\omega \circ f) = \gamma(\omega) deg(f) = n$. Thus γ is an epimorphism. □

(B) **Corollary.** (i) Each one-dimensional framed submanifold (M^1, U) of E^3 is frame-cobordant to (S^1, V_r), for some $r \in \mathbb{Z}$.

Proof. By the theorem, $\gamma(M^1, U) = r = \gamma(S^1, V_r)$ implies $(M^1, U) \sim (S^1, V_r)$.

(ii) For any framed $(M^1, W) \subset E^{n+1}$, we have $(M^1, W) \sim \mathbb{E}^{n-2}(S^1, V_r)$ for some $r \in \mathbb{Z}$. ($\mathbb{E}^{n-2}$ denotes iterated suspension.)

Proof. First use the fact any framed submanifold of E^{n+1} is fr-cobordant to a connected one: $(M^1, W) \sim (S^1, U)$. This is a consequence of part (i).

Recall now that, by the Freudenthal suspension theorem (Theorem 11), $\mathbb{E} : \Pi^k_n \to \Pi^k_{n+1}$ is an epimorphism if $n \geq k + 1$. In particular, $\mathbb{E} : \Pi^1_n \to \Pi^1_{n+1}$ is *epi*, if $n \geq 2$. Iterating, we find that: $\mathbb{E}^{n-2} : \Pi^1_2 \to \Pi^1_n$ is an epimorphism. The conclusion now follows from part (i). □