CURVATURE UNDER CONFORMAL CHANGE OF METRIC

We consider a conformal change $\tilde{g} = e^{2f}g$. From the usual formula for Christoffel symbols, we find:

$$\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + f_j\delta_i^k + f_i\delta_j^k - f^k g_{ij},$$

which translates to:

$$\widetilde{\nabla}_X Y = \nabla_X Y + T(X,Y),$$

where T is the symmetric $(2,1)$ tensor:

$$T(X,Y) = (Xf)Y + (Yf)X - \langle X,Y\rangle\nabla f.$$

Turning to the (3,1) curvature $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, we find (by iterating the above, and after cancellations):

$$\tilde{R}(X,Y)Z = R(X,Y)Z + (\nabla_X T)(Y,Z) - (\nabla_Y T)(X,Z) + T(X,T(Y,Z)) - T(Y,T(X,Z)).$$

Expressed in terms of f, the second-order terms are (with Hf the Hessian):

$$(\nabla_X T)(Y,Z) = Hf(X,Y)Z + Hf(X,Z)Y - \langle Y,Z\rangle\nabla_X\nabla f,$$

$$(\nabla_X T)(Y,Z) - (\nabla_Y T)(X,Z) = Hf(X,Z)Y - Hf(Y,Z)X - \langle Y,Z\rangle\nabla_X\nabla f + \langle X,Z\rangle\nabla_Y\nabla f.$$

For the quadratic first-order terms, we find:

$$\begin{aligned} &T(X,T(Y,Z)) - T(Y,T(X,Z)) \\ &= (Yf)(Zf)X - (Xf)(Zf)Y + \langle Y,Z\rangle(Xf)\nabla f - \langle X,Z\rangle(Yf)\nabla f \\ &\quad -|\nabla f|^2(\langle Y,Z\rangle X - \langle X,Z\rangle Y). \end{aligned}$$

The symmetries in this expression become evident when we consider the (4,0) curvature, taking g-inner product with W:

$$\begin{aligned} &e^{-2f}\tilde{R}(X,Y,Z,W) = \langle R(X,Y)Z,W\rangle \\ &-Hf(X,W)\langle Y,Z\rangle + Hf(Y,W)\langle X,Z\rangle - \langle X,W\rangle Hf(Y,Z) + \langle Y,W\rangle Hf(Y,Z) \\ &+\langle X,W\rangle(Yf)(Zf) - \langle Y,W\rangle(Xf)(Zf) + (Xf)(Wf)\langle Y,Z\rangle - (Yf)(Wf)\langle X,Z\rangle \\ &-|\nabla f|^2(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle). \end{aligned}$$

The second line involves the symmetric (2,0) tensor defined as:

$$(df\circ df)(X,Y) = (Xf)(Yf).$$

Now recall the definition of the Kulkarni-Nomizu product, which associates to two symmetric 2-tensors h,k an algebraic (4,0) curvature tensor, in a symmetric way:

$$(h \owedge k)(X,Y,Z,W) = h(X,W)k(Y,Z) - h(Y,W)k(X,Z) + k(X,W)h(Y,Z) - k(Y,W)h(X,Z).$$

We see immediately that the conformally changed (4,0) curvature is:

$$e^{-2f}\tilde{R} = R - Hf \owedge g + (df\circ df) \owedge g - |\nabla f|^2 \frac{g \owedge g}{2}$$

$$= R - [Hf - (df\circ df) + (1/2)|df|^2 g] \owedge g.$$

The Ricci and Scalar traces of a K-N product (w.r.t. g) are easily computed:

$$Ric_g(h \owedge k) = (tr_g h)k + (tr_g k)h - (h\bullet k + k\bullet h),$$

$$Scal_g(h \owedge k) = (tr_g h)(tr_g k) - \langle h, k\rangle_g,$$

where $(h\bullet k)(X,Y) = \sum_i h(X,e_i)k(e_i,Y)$, (e_i) g-o.n. In particular, when one of the bilinear forms is the metric:

$$Ric_g(h \owedge g) = (n-2)h + (tr_g h)g,$$

$$Scal_g(h \owedge g) = 2(n-1)tr_g h.$$

Specializing to the case on hand: note the o.n. frame for $\tilde{g}$ corresponding to (e_i) is $e^{-f}e_i$, so:

$$\widetilde{Ric}(X,Y) = e^{-2f}\sum_i \tilde{R}(X,e_i,e_i,Y) = Ric_g(e^{-2f}\tilde{R})(X,Y),$$

and $tr_g[Hf - (df\circ df) + (1/2)|df|^2 g] = \Delta f + \frac{n-2}{2}|df|^2$ so:

$$\widetilde{Ric} = Ric - (n-2)[Hf - (df\circ df) + \frac{1}{2}|df|^2 g] - [\Delta f + \frac{n-2}{2}|df|^2]g$$

$$\widetilde{Ric} = Ric - (n-2)[Hf - (df\circ df)] - [\Delta_g f + (n-2)|df|^2]g,$$

a surprisingly simple formula (cp. [LP87], (2.5)). Turning to the scalar curvature, we have:

$$\tilde{S} = \sum_j e^{-2f}\widetilde{Ric}(e_j,e_j) = e^{-2f}\{S_g - (n-2)[\Delta f + (n-1)|df|^2] - n\Delta f\},$$

$$\tilde{S} = e^{-2f}[S_g - 2(n-1)\Delta f - (n-1)(n-2)|df|^2].$$

(Cf. [LP87], (2.6).) Making the change of variable $e^{2f} = u^{\frac{4}{n-2}}$ (where $u > 0$), we find:

$$\tilde{S} = u^{-\frac{4}{n-2}}(S_g - a_n\frac{\Delta u}{u}), \quad a_n = \frac{4(n-1)}{n-2},$$

$$\tilde{S} = u^{-\frac{n+2}{n-2}}(S_g u - a_n\Delta u) = u^{-\frac{n+2}{n-2}}L_g u,$$

where $L_g u = -a_n\Delta_g u + S_g u$ is the *conformal Laplacian* of g.

For $\tilde{g} = u^{4/n-2}g$, the Ricci curvature is:

$$\widetilde{Ric} = Ric - 2\frac{Hu}{u} + \frac{2n}{n-2}\frac{du\circ du}{u^2} - \frac{2}{n-2}(\frac{\Delta u}{u} + \frac{|du|^2}{u^2})g.$$

We can eliminate the $df \circ df$ term in the expression for $\widetilde{Ric}$ by writing the new metric in the form $\tilde{g} = \phi^{-2} g$ (so $e^{2f} = \phi^{-2}$). We find:

$$\widetilde{Ric} = Ric + \phi^{-1}[(n-2)H\phi - (n-1)\frac{|d\phi|_g^2}{\phi} g + (\Delta\phi) g].$$

And for the trace-free Ricci tensors:

$$\widetilde{Ric}^{tf} = Ric^{tf} + (n-2)\frac{(H\phi)^{tf}}{\phi}.$$

Example: metrics conformal to the euclidean metric in R^n, polar coordinates (r, ω), $u = u(r)$.

With $ds^2 = dr^2 + r^2 d\omega^2$, for the Hessian we have for the euclidean metric (where ∂_ω is a tangent vector to S^{n-1}, $|\partial_\omega| = r$):

$$Hu(\partial_r, \partial_r) = u''(r), \quad Hu(\partial_r, \partial_\omega) = 0, \quad Hu(\partial_\omega, \partial_\omega) = ru'.$$

And for the Ricci curvature of $\tilde{g}$, we have from the above:

$$\widetilde{Ric}(\partial_r, \partial_r) = -\frac{2(n-1)}{n-2}(\frac{u''}{u} + \frac{u'}{ru} - \frac{(u')^2}{u^2}), \quad \widetilde{Ric}(\partial_r, \partial_\omega) = 0,$$

$$\widetilde{Ric}(\frac{\partial_\omega}{r}, \frac{\partial_\omega}{r}) = -\frac{2}{n-2}(\frac{u''}{u} + (2n-3)\frac{u'}{ru} + \frac{(u')^2}{u^2}).$$

As a check, we find as expected:

$$u^{4/n-2}\tilde{S} = \widetilde{Ric}(\partial_r, \partial_r) + \sum_i \widetilde{Ric}(\frac{\partial_{\omega_i}}{r}, \frac{\partial_{\omega_i}}{r}) = -a_n(\frac{u''}{u} + \frac{n-1}{r}\frac{u'}{u}).$$

Example: Spatial Schwarzschild metric. The one-parameter family of n-dimensional spatial Schwarzschild metrics is given in original coordinates as the metric outside a ball in R^n (with polar coordinates (s, ω)):

$$g_m = \frac{ds^2}{1 - \frac{2m}{s^{n-2}}} + s^2 d\omega^2, \quad s > (2m)^{1/n-2}.$$

The change of coordinates to 'isotropic coordinates' (r, ω) expresses g_m in conformally flat form:

$$g_m = u(r)^{4/n-2}(dr^2 + r^2 d\omega^2), \quad u(r) = 1 + \frac{m}{2r^{n-2}}, \quad s = r(1 + \frac{m}{2r^{n-2}})^{2/n-2}, \quad r > (m/2)^{1/n-2}.$$

Exercise: Check that this coordinate change $s = s(r)$ indeed changes the form of the metric g_m as described. *Hint:* The coordinate change is suggested by the $d\omega^2$ part of the metric, so what needs to be checked is:

$$\frac{ds}{(1 - \frac{2m}{s^{n-2}})^{1/2}} = (1 + \frac{m}{2r^{n-2}})^{2/n-2} dr.$$

Exercise: Use the formulas derived above to compute the expressions for the Ricci curvature of g_m:

$$Ric(\partial_r, \partial_r) = -\frac{m(n-1)(n-2)}{r^n}(1 + \frac{m}{2r^{n-2}})^{-2}, \quad Ric(\partial_r, \partial_\omega) = 0,$$

$$Ric(\frac{\partial_\omega}{r}, \frac{\partial_\omega}{r}) = \frac{m(n-2)}{r^n}(1 + \frac{m}{2r^{n-2}})^{-2}.$$

As a check, confirm that (as expected from $\Delta_0 u = 0$), the scalar curvature vanishes:

$$S = u^{-4/n-2}(Ric(\partial_r, \partial_r) + \sum_i Ric(\frac{\partial_{\omega_i}}{r}, \frac{\partial_{\omega_i}}{r})) = 0.$$

Example: the ADM mass of spatial Schwarzschild.

$$g_m = \frac{dr^2}{1 - \frac{2m}{r^{n-2}}} + r^2 d\omega^2 = (1 + \frac{2m}{r^{n-2}} + O_2(r^{-n+1}))dr^2 + r^2 d\omega^2,$$

So the difference with the euclidean metric is:

$$e = e_{ij} dx^i dx^j = (\frac{2m}{r^{n-2}} + O_2(r^{-n+1}))dr^2, \quad e_{ij} = 2m\frac{x^i x^j}{r^n} + O_2(r^{1-n}),$$

since $dr^2 = \frac{x^i x^j}{r^2} dx^i dx^j$.

The mass integrand 1-form is $\mu = (e_{ij,i} - e_{ii,j})dx^j$ (with summation convention). We compute the first partial derivatives:

$$e_{ij,k} = \frac{2m}{r^n}(\delta_{ik} x^j + \delta_{jk} x^i - n\frac{x^i x^j x^k}{r^2}) + O(r^{-n})$$

This gives for the mass 1-form:

$$\mu = \frac{2m(n-1)}{r^n} x^i dx^i + O(r^{-n}).$$

So the surface integral in the definition of ADM mass is (with $\nu^i = x^i/r$):

$$\oint_{S_r} \mu[\nu] d\sigma_r = 2m(n-1) \oint_{S_r} (\frac{x^i x^i}{r^{n+1}} + O(r^{-n}))r^{n-1} d\omega.$$

So we see that:

$$\lim_{r \to \infty} \oint_{S_r} \mu[\nu] d\sigma_r = 2m(n-1)\omega_{n-1}$$

(ω_{n-1} is the area of the $(n-1)$-sphere, so $\omega_2 = 4\pi$.). This explains the normalization constant in the definition of m_{ADM}. For the metrics g_m:

$$m_{ADM}(g_m) = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r \to \infty} \oint_{S_r} \mu[\nu] d\sigma_r = m.$$

Example: Obata's Theorem: A conformal metric g on S^n of constant scalar curvature must have constant (sectional) curvature, and hence be isometric to the standard sphere (up to scaling).

Proof. Write $g_{can} = \phi^{-2}g$, where g_{can} is the standard metric. Then, denoting by $B = Ric - \frac{1}{n}Sg$ the trace-free Ricci tensor, which vanishes for the standard metric, we find:

$$0 = B + (n-2)\phi^{-1}(H\phi - \frac{\Delta\phi}{n}g).$$

Integrating over S^n:

$$\int_{S^n} \phi|B|_g^2 dV_g = -(n-2)\int_{S^n} \langle B, H\phi - \frac{\Delta\phi}{n}g\rangle dV_g = -(n-2)\int_{S^n} \langle B, H\phi\rangle dV_g,$$

since B is trace-free. Since the scalar curvature S_g is constant, the contracted second Bianchi identity:

$$dS - 2\delta(Ric) = 0$$

implies $\delta(Ric) = 0$, and hence also $\delta B = 0$. But letting α be the 1-form $\alpha(X) = B(X, \nabla\phi)$, we find:

$$\delta\alpha = (\delta B)(\nabla\phi) + \langle B, H\phi\rangle.$$

Thus the last integral vanishes, so $B \equiv 0$. Since the Weyl tensor of g also vanishes, we find that g is a constant curvature metric, hence necessarily the standard metric on S^n, up to scaling.