NOTES ON THE YAMABE PROBLEM

These are course notes to accompany the survey by J. Lee and T.Parker ([BAMS, July 1987]), of which they are a very rough summary.

1. The Yamabe invariant of a conformal class.

The Yamabe functional on Riemannian metrics on a compact manifold $M^n, n \geq 3$, is defined by:

$$Q(g) = \frac{\int_M S_g dV_g}{Vol(M,g)^{\frac{n-2}{n}}}.$$

The power in the denominator makes it scaling-invariant. Here we consider this functional within a conformal class of metrics $\tilde{g} = u^{4/n-2}g$, for a given background metric g and arbitrary positive smooth function u on M:

$$Q_g(u) = Q(\tilde{g}) = \frac{\int_M uLudV_g}{(\int_M u^{2n/n-2}dV_g)^{\frac{n-2}{n}}} = \frac{E[u]}{||u||_p^2},$$

where $Lu = -a\Delta_g u + S_g u$ (the conformal Laplacian of g, $a = 4(n-1)/(n-2)$), $E(u) = \int_M (a|\nabla u|_g^2 + S_g u^2)dV_g$ the associated quadratic form, and $p = 2n/n-2$.

Computing the first variation with $u_t = u + tv$, we find:

$$\frac{d}{dt}_{|t=0} Q_g(u_t) = \frac{2}{||u||_p^2}\int_M (Lu)vdV_g - 2\frac{E(u)}{||u||_p^{p+2}}\int_M u^{\frac{n+2}{n-2}}vdV_g$$

$$= \frac{2}{||u||_p^2}\int_M [Lu - \lambda u^{\frac{n+2}{n-2}}]vdV_g, \quad \lambda = \frac{E(u)}{||u||_p^p}.$$

Thus u is a critical point for Q_g iff:

$$Lu = \lambda u^{\frac{n+2}{n-2}}, \quad \lambda = \frac{E(u)}{||u||_p^p}.$$

This means the metric $\tilde{g}$ has constant scalar curvature $S_{\tilde{g}} \equiv \lambda$. In fact note:

$$\lambda = \frac{\int_M (Lu)udV_g}{||u||_p^p} = \frac{\int_M S_{\tilde{g}} u^{\frac{n+2}{n-2}}udV_g}{vol(M,\tilde{g})} = \frac{\int_M S_{\tilde{g}} dV_{\tilde{g}}}{vol(M,\tilde{g})},$$

the mean value of scalar curvature over M in the metric $\tilde{g}$, as expected.

The Sobolev inequality on (M,g) corresponding to the embedding $L_1^2 \subset L^p$:

$$||\phi||_p^2 \leq C(||\nabla\phi||_2^2 + ||\phi||_2^2) \text{ for all } \phi \in C^\infty(M),$$

with $C = C(M,g)$, guarantees the functional Q_g is bounded below:

$$Q_g(u) = \frac{E(u)}{||u||_p^2} \geq \frac{||\nabla u||_2^2}{||u||_p^2} - (\sup_M |S_g|)\frac{||u||_2^2}{||u||_p^2} \geq \frac{1}{C} - [(\sup_M |S_g|) + 1]vol(M,g)^{2/n},$$

by Hölder's inequality. Thus to find critical points (which would give metrics of constant scalar curvature in the same conformal class) it is natural to look for minimizers of Q_g. Define the *Yamabe invariant* of g by:

$$\lambda(M,g) = \inf\{Q_g(u); u \in C^\infty(M)\}.$$

Remark. This is in fact an invariant of the conformal class of g. Indeed, let $\bar{g} = w^{4/n-2}g$. Then for any $\phi \in C^\infty(M)$:

$$|\nabla\phi|^2_{\bar{g}} = w^{-4/n-2}|\nabla\phi|^2_g, \quad S_{\bar{g}}\phi^2 dV_{\bar{g}} = (-a\Delta_g w + S_g w)w\phi^2 dV_g.$$

$$(|\nabla\phi|^2_{\bar{g}} + S_{\bar{g}}\phi^2)dV_{\bar{g}} = (aw^2|\nabla\phi|^2_g - a(\Delta_g w)w\phi^2 + S_g w^2\phi^2)dV_g$$

$$E(\phi,\bar{g}) = \int_M (aw^2|\nabla\phi|^2_g + a|\nabla w|^2_g\phi^2 + 2a\langle\nabla w, \nabla\phi\rangle_g w\phi + S_g w^2\phi^2)dV_g$$

$$= \int_M (a|w\nabla\phi + \phi\nabla w|^2 + S_g w^2\phi^2)dV_g = E(w\phi, g).$$

And since $||\phi w||_{L^p(g)} = ||\phi||_{L^p(\bar{g})}$, it follows that:

$$Q_{\bar{g}}(\phi) = Q_g(\phi w),$$

and this implies:

$$\lambda(M,g) = \lambda(M,\bar{g}).$$

2. The case of S^n.

Let $\sigma : S^n \setminus \{P\} \to R^n$ be stereographic projection, $\rho : R^n \to S^n \setminus \{P\}$ its inverse. The canonical metric g_{can} pulls back under ρ to the incomplete metric of constant scalar curvature $n(n-1)$ in R^n:

$$g_1 = \rho^* g_{can} = \frac{4}{(1+|x|^2)^2}dx^2 = 4u_1^{4/n-2}dx^2, \quad u_1(x) = \frac{1}{(1+|x|^2)^{n-2/2}}.$$

By Liouville's theorem, the group of conformal diffeomorphisms of S^n is a finite-dimensional Lie group, generated (via conjugation by ρ) by orthogonal transformations, translations and homotheties of R^n, the latter being the maps $\delta_\alpha(x) = \frac{x}{\alpha}, \alpha > 0$. We have:

$$g_\alpha = \delta^*_\alpha(g_1) = \frac{4}{(1+\frac{|x|^2}{\alpha^2})^2}d(\frac{x}{\alpha})^2 = \frac{4\alpha^2}{(\alpha^2+|x|^2)^2}dx^2 = 4u_\alpha^{4/n-2}dx^2,$$

where:

$$u_\alpha(x) = (\frac{\alpha}{\alpha^2+|x|^2})^{(n-2)/2}.$$

For the scalar curvature, we have:

$$S(u_\alpha^{4/n-2}dx^2) = S(\frac{1}{4}g_\alpha) = 4S(g_\alpha) = 4n(n-1).$$

Thus the euclidean Laplacian of u_α satisfies:

$$-a\Delta_0 u_\alpha = 4n(n-1)u_\alpha^{\frac{n+2}{n-2}}.$$

The family of conformal metrics of constant scalar curvature on S^n is not large, and described by the following *uniqueness theorem*:

Theorem [Obata 1971]. (See [L-P prop 3.1].) Any metric of constant scalar curvature on S^n pointwise conformal to g_{can} is obtained from g_{can} by scaling, or by pullback via a conformal diffeomorphism of S^n. By pullback via ρ, this describes the set of all positive solutions in R^n of:

$$-a\Delta_0 u = \lambda u^{\frac{n+2}{n-2}}$$

for some constant $\lambda \in R$: λ must be positive, and they are the u_α, their constant multiples and compositions with translations and orthogonal transformations (acting on the independent variable x.)

We conclude that this is also a complete list of possible critical points of the Yamabe functional in R^n:

$$Q(u) = \frac{\int_{R^n} a|\nabla u|_0^2 d^n x}{||u||_p^2}, \quad u \in L_1^2(R^n),$$

Of course, the u_α are critical points; but at this point we don't know if they are minimizers. We need an independent (and harder to prove) *existence theorem:*

Theorem. (Prop 4.6 in [L-P], proof attributed to K. Uhlenbeck). There exists a positive smooth function ϕ on S^n minimizing the Yamabe functional:

$$Q_{g_{can}}(\phi) = \lambda(S^n, g_{can}).$$

(And then, from uniqueness for critical points, we do know the complete set of minimizers.)

Connection with the Sobolev inequality. In R^n we have the following estimate, corresponding to the critical Sobolev embedding $L_1^2 \subset L^p$:

$$||\phi||_p^2 \leq \sigma_n ||\nabla\phi||_2^2, \quad \forall \phi \in C_c^\infty(R^n)$$

(i.e. smooth, of compact support.) In fact we let $\sigma_n > 0$ be the optimal constant, so:

$$\frac{1}{\sigma_n} = \inf\{\frac{\int_{R^n} |\nabla\phi|^2 d^n x}{||\phi||_p^2}; \phi \in C_c^\infty(R^n)\}.$$

We see (using density of $C_c^\infty(R^n)$ in $L_1^2(R^n)$) that $\frac{1}{\sigma_n} = \frac{1}{a}\lambda(S^n, g_{can})$, and from the two results just quoted we know this infimum is achieved by (pullbacks to R^n of) the standard metric g_{can} on S^n, where we know its value ($\omega_n = vol(S^n)$):

$$\Lambda := \lambda(S^n, [g_{can}]) = Q_{g_{can}}(1) = \frac{E_{g_{can}}(1)}{vol(S^n)^{2/p}} = n(n-1)\omega_n^{2/n}, \quad \sigma_n^{-1} = \frac{\Lambda}{a}.$$

It is a useful fact that the corresponding Sobolev inequality on a compact Riemannian manifold holds with essentially the same constant:

Theorem. ([T. Aubin 1976], Thm 2.3 in [L-P]). Let σ_n be the best constant in the Sobolev inequality in R^n, and let (M^n, g) be any compact Riemannian manifold. Then for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ so that for all $\phi \in C^\infty(M)$:

$$||\phi||_p^2 \leq (1+\epsilon)\sigma_n \int_M |\nabla\phi|_g^2 dV_g + C_\epsilon \int_M \phi^2 dV_g, \quad p = \frac{2n}{n-2}.$$

3. $\lambda(M)$ is always bounded above by $\lambda(S^n)$.

Theorem. Let (M, g) be a compact Riemannian manifold, $n \geq 3$. Then $\lambda(M, [g]) \leq \lambda(S^n, [g_{can}]) := \Lambda$.

Proof. (See [L-P], Lemma 3.4.) We need a suitable 'test function' for the Yamabe quotient on M. The idea is to 'transplant' the u_α from R^n to M, for α small. (This corresponds to grafting on M a *huge* sphere, so the effect of M itself becomes negligible.) But first we have to deal with the fact the u_α do not have compact support; so still in R^n, we fix an $\epsilon > 0$ and multiply it by a smooth 'bump function' η, with support in $B_{2\epsilon}$ and equal to 1 on B_ϵ. That is, let $\phi = \eta u_\alpha$, where both are radial functions. Then:

$$\int_{R^n} a|\nabla\phi|^2 d^n x \leq \int_{B_{2\epsilon}} a|\partial_r u_\alpha|^2 d^n x + C\int_{A_\epsilon} (u_\alpha|\partial_r u_\alpha| + u_\alpha^2) d^n x.$$

Remark. There are two parameters, and the idea is to choose the cutoff parameter ϵ small (depending only on the geometry of (M, g), considered fixed), then choose α small depending on ϵ. Thus in this proof it is not important to keep track of the dependence of the estimates on ϵ.

About the radial functions $u_\alpha(x) = (\frac{\alpha}{\alpha^2+r^2})^{(n-2)/2}$ we know everything:

$$a||\nabla u_\alpha||_2^2 = \Lambda||u_\alpha||_p^2, \quad 0 < u_\alpha \leq \alpha^{\frac{n-2}{2}} r^{2-n}, \quad |\partial_r u_\alpha| \leq (n-2)\alpha^{\frac{n-2}{2}} r^{1-n}.$$

So for the annular term (treated as an 'error'):

$$\int_{A_\epsilon} (u_\alpha|\partial_r u_\alpha| + u_\alpha^2) d^n x \leq C\alpha^{n-2}$$

(where C depends on ϵ). For the main term:

$$\int_{R^n} a|\partial_r u_\alpha|^2 d^n x = \Lambda(\int_{B_\epsilon} u_\alpha^p d^n x + \int_{B_\epsilon^c} u_\alpha^p d^n x)^{2/p}$$

$$\leq \Lambda(\int_{B_{2\epsilon}} \phi^p d^n x + \int_{B_\epsilon^c} \alpha^n r^{-2n} d^n x)^{2/p} \leq \Lambda(\int_{B_{2\epsilon}} \phi^p d^n x)^{2/p} + O(\alpha^n).$$

To complete the estimate, at this point we need to divide by $||\phi||_p^2$. This is fine for the main term, but since there are error terms we also need a lower bound on the denominator, a point left implicit in [L-P]. To see this is not a problem, recall $Vol(R^n, g_\alpha) = \omega_n$, so:

$$\int_{R^n} u_\alpha^p d^n x = Vol(R^n, u_\alpha^{4/n-2} dx^2) = \frac{\omega_n}{4^n},$$

a constant independent of α. In addition,

$$u_\alpha(x) = \frac{1}{\alpha^{(n-2)/2}} u_1(\frac{x}{\alpha}) \Rightarrow \int_{B_\epsilon} u_\alpha^p(x) d^n x = \int_{B_{\frac{\epsilon}{\alpha}}} u_1^p(y) d^n y,$$

making the obvious change of variable $x = \alpha y$. So the way to guarantee a lower bound on the denominator (independent of α) is to choose ϵ small, then α small so that:

$$\int_{B_{\frac{\epsilon}{\alpha}}} u_1^p(y) d^n y \geq \frac{1}{2} \int_{R^n} u_1^p(y) d^n y.$$

So we find for the Yamabe quotient of $\phi = \eta u_\alpha$ (for the euclidean metric):

$$Q_0(\phi) = \frac{E_0(\phi)}{||\phi||_p^2} \leq \Lambda + C\alpha^{n-2}.$$

To obtain the estimate on a compact manifold (M, g), we choose 2ϵ to be smaller than the injectivity radius, and transfer ϕ to a normal ball at some point $p \in M$, via normal coordinates at p. To control the error terms, we observe that $dV_g = (1 + O(r))d^n x$ and $g^{rr} = 1$ in normal coordinates, so $|\nabla\phi|_g^2 = |\partial_r \phi|^2$ as before. Then for the energy we have:

$$E_g(\phi) = \int_M (a|\nabla\phi|_g^2 + S_g\phi^2) dV_g \leq (1 + C\epsilon)(\Lambda||\phi||_p^2 + C\alpha^{n-2} + C\int_0^{2\epsilon} u_\alpha^2(r) r^{n-1} dr).$$

The last term (which comes from the scalar curvature) is a one-variable integral, and a 'calculus lemma' (Lemma 3.5 in [L-P]) yields that it is bounded by a constant times α. Thus, appealing again to the remark above concerning the denominator, we obtain:

$$Q_g(\phi) \leq (1 + C\epsilon)(\Lambda + C\alpha),$$

which is enough to yield the conclusion $\lambda(M, [g]) \leq \Lambda$.

4. Existence for the subcritical problem.

Instead of solving the critical variational problem directly, Yamabe's approach was to consider subcritical variational problems:

$$\lambda_s(M) = \{\min Q_s(\phi) = \frac{E(\phi)}{||\phi||_s}; \phi \in C^\infty(M)\}, \text{ where } 2 \leq s < p;$$

'Subritical' is in the sense of the Sobolev embedding $W^{1,2} \subset L^s$:

$$||\phi||_s^2 \leq \sigma_s(\int_M |\nabla\phi|^2 dV_g + \int_M \phi^2 dV_g),$$

which implies the quotient Q_s is bounded below (possibly by a negative constant.) The Euler-Lagrange equation for normalized critical points ($||\phi||_s = 1$) is:

$$L\phi = \lambda_s \phi^{s-1}, \quad L\phi = -a\Delta_g\phi + S_g\phi, a = \frac{4(n-1)}{n-2}, \lambda_s = \lambda_s(M) \qquad (\mathbb{Y}_s)$$

This equation has smooth positive solutions, obtained by the direct method. The main reason is the fact the embedding $W^{1,2} \subset L^s$ is *compact* if $s < p$ (that is, bounded sequences in $W^{2,1}$ have subsequences converging strongly in L^s norm.) The following regularity lemma is used:

Lemma. Suppose $\phi \in W^{1,2}$ is a nonnegative weak solution of $\mathbb{Y}_s$, where $2 \leq s \leq p$. Assume $|\lambda_s| \leq K$. If $\phi \in L^r$ for some $r > (s-2)\frac{n}{2}$, then ϕ is smooth, either $\phi > 0$ or vanishes identically, and $||\phi||_{C^{2,\alpha}} \leq C$ (with C depending on $||\phi||_r$ and K.)

Proof. Writing $-a\Delta\phi = \lambda_s\phi^{s-1} - S\phi$, we see if $\phi \in L^r$, the right-hand side is in L^q, $q = \frac{r}{s-1}$. By elliptic regularity in Sobolev spaces, we have $\phi \in W^{2,q}$. By Sobolev embedding, $W^{2,q} \subset L^{r'}$, where:

$$\frac{1}{r'} = \frac{1}{q} - \frac{2}{n} = \frac{s-1}{r} - \frac{2}{n} = \frac{ns - n - 2r}{nr} < \frac{1}{r},$$

the last inequality being equivalent to $r > (s-2)\frac{n}{2}$. Thus $r' > r$, and we've gained some integrability. (In fact if $r > (s-1-C^{-1})\frac{n}{2}$ for some $C > 1$, as is the case under the hypothesis on r, we have $r' > Cr$.) Iterating this argument, we find $\phi \in W^{2,q}$ for any $q > 1$. By Sobolev embedding into Hölder spaces, we know $\phi \in C^\alpha$ as soon as $q > \frac{n}{2}$, where $0 < \alpha < 1$ and $\frac{1}{q} < \frac{2-\alpha}{n}$. And then also $\phi^{s-1} \in C^\alpha$, so by elliptic regularity in Hölder spaces $\phi \in C^{2,\alpha}$.

As for positivity, let $m > 0$ satisfy $m > sup_M(S - \lambda_s\phi^{s-2})$. Then, since $\phi \geq 0$:

$$-a\Delta\phi + m\phi = m\phi - S\phi + \lambda_s\phi^{s-1} \geq 0,$$

so the strong maximum principle implies either $\phi > 0$ on M, or $\phi \equiv 0$ on M. Thus $\phi^{s-1} > 0$ on M, and then repeated application of elliptic regularity yields $\phi \in C^\infty$. □

Theorem: minimizers for the subcritical problem. [Yamabe 1960]. For $2 \leq s < p$, there exists a smooth positive solution ϕ_s to $\mathbb{Y}_s$, with $Q_s(\phi_s) = \lambda_s(M)$ and $||\phi_s||_s = 1$.

Proof. We may assume $vol(M, g) = 1$. Let $u_i \in C^\infty$ be a smooth minimizing sequence for Q_s with $||u_i||_s \leq 1$. Since $E(|u_i|) \leq E(u_i)$, we may assume $u_i \geq 0$.

It's easy to see (u_i) is bounded in $W^{1,2}$:

$$||u_i||_{W^{1,2}} = int_M(|\nabla u_i|^2 + u_i^2)dV_g = \frac{1}{a}E(u_i) + \int_M (1 - \frac{S}{a})u_i^2 dV_g \leq \frac{1}{a}E(u_i) + C||u_i||_2^2,$$

while $||u_i||2 \leq ||u_i||_s = 1$, by Hölder's inequality (since $vol(M) = 1$.)

Since the embedding $W^{1,2} \subset L^s$ is compact, a subsequence converges weakly in $W^{1,2}$, strongly in L^s and pointwise a.e. to a function $\phi_s \geq 0$, with $||\phi_s||_s = 1$.

Since convergence in L^s implies convergence in L^2, we have: $\int_M Su_i^2 dV_g \to \int_M S\phi^2 dV_g$. By weak convergence in $W^{1,2}$:

$$\int_M |\nabla \phi_s|^2 dV_g = \lim \int_M \langle \nabla \phi_s, \nabla u_i \rangle dV_g \leq \limsup (\int_M |\nabla u_i|^2 dV_g)^{1/2} (\int_M |\nabla \phi_s|^2 dV_g)^{1/2}.$$

It then follows that:

$$Q_s(\phi_s) = E(\phi_s) \leq \lim E(u_i) = \lambda_s(M),$$

so ϕ_s is a minimizer, hence a nonnegative weak solution of $\mathbb{Y}_s$. By the earlier elliptic regularity lemma, ϕ_s is positive and smooth. $\square$

The next lemma has a simple proof (See Lee-Parker, Lemma 4.3]).

Lemma. (Behavior of λ_s.) Assume $vol(M, g) = 1$. Then $|\lambda_s|$ is nonincreasing, for $s \in [2, p]$. If $\lambda_s < 0$ for some s, then this is true for all s (and $\lambda(M) < 0$.) If $\lambda(M) \geq 0$, then λ_s is left-continuous as $s \uparrow p$.

5. Solution of the critical Yamabe equation.

The idea is to take limits of the subcritical solution ϕ_s, as $s \uparrow p$. This requires a uniform bound, which will follow from the following lemma (see [Trudinger 68],[Aubin 76]).

Lemma. Assume $\lambda(M) < \lambda(S^n) := \Lambda$. For $2 \leq s < p$, let ϕ_s be a smooth positive solution of the subcritical Yamabe equation $\mathbb{Y}_s$, normalized so that $||\phi_s||_s = 1$. There exist $s_0 \in [2, p)$, $r > p$ and $C > 0$ so that for all $s \geq s_0$ we have $||\phi_s||_r \leq C$.

Proof. We may assume $vol(M, g) = 1$, so Hölder's inequality implies the L^q norm of a function is a nondecreasing function of q. Let $w = \phi_s^{1+\delta}$. The goal is to show $||w||_p^2 \leq C||w||_2^2$. Since $||w||_p = ||\phi_s||_{p(1+\delta)}^{1+\delta}$, we would then let $r = p(1 + \delta)$ and observe that, assuming $s \geq s_0 = 2(1 + \delta)$:

$$||w||_2 = ||\phi_s||_{2(1+\delta)}^{1+\delta} \leq ||\phi_s||_s^{1+\delta} = 1,$$

giving a bound on $||\phi_s||_{p(1+\delta)}$ independent of s in the range $[s_0, p]$.

Multiplying $L\phi_s = \lambda_s \phi_s^{s-1}$ by $\phi_s^{1+2\delta}$ and integrating, we find:

$$a(1 + 2\delta) \int_M \langle d\phi_s, \phi_s^{2\delta} d\phi_s \rangle dV_g = \lambda_s \int_M \phi_d^{s+2\delta} dV_g - \int_M S\phi_s^{2+2\delta} dV_g,$$

or in terms of w, with $dw = (1+\delta)\phi_s^\delta d\phi_s$:

$$a\frac{1+2\delta}{(1+\delta)^2}\int_M |dw|^2 dV_g = \int_M (\lambda_s w^2 \phi_s^{s-2} - Sw^2)dV_g.$$

Now recall the critical Sobolev inequality on M, with the near-optimal constant $(1+\epsilon)\sigma_n$, $\sigma_n = \frac{a}{\Lambda}$:

$$||w||_p^2 \leq (1+\epsilon)\frac{a}{\Lambda}\int_M |dw|^2 dV_g + C_\epsilon \int_M w^2 dV_g,$$

which combined with the above gives:

$$||w||_p^2 \leq (1+\epsilon)\frac{(1+\delta)^2}{1+2\delta}\frac{\lambda_s}{\Lambda}\int_M w^2\phi_s^{s-2}dV_g + C'_\epsilon ||w||_2^2.$$

If $\lambda_p = \lambda(M) < 0$, since λ_s is increasing in s it follows $\lambda_s < 0$, and then the desired estimate $||w||_p^2 \leq C||w||_s^2$ follows immediately. So from this point on assume $\lambda(M) \geq 0$. The remaining integral can be estimated via Hölder's inequality:

$$\int_M w^2 \phi_s^{s-2} dV_g \leq ||w||_p^2 ||\phi_s||_{(s-2)\frac{n}{2}}^{s-2}.$$

(To see this, let $f = w^2, g = \phi_s^{s-2}$. Since we want to use the L^p norm in the estimate of w, let $q = \frac{p}{2}$ be the exponent for f, and consequently, with $\frac{1}{q'} = 1 - \frac{2}{p} = \frac{2}{n}$, $q' = \frac{n}{2}$ will be the exponent for g in:

$$\int_M fg dV_g \leq ||f||_q ||g||_{q'} = ||w^2||_{\frac{p}{2}} ||\phi_s^{s-2}||_{\frac{n}{2}} = ||w||_p^2 ||\phi_s||_{(s-2)\frac{n}{2}}^{s-2},$$

as claimed.)

It is easy to see that $s \leq p$ is equivalent to $(s-2)\frac{n}{2} \leq s$, so $||\phi_s||_{(s-2)\frac{n}{2}} \leq ||\phi_s||_s = 1$ (again by Hölder, since $vol(M) = 1$.) Since $\lambda(M) \geq 0$ (and consequently $\lambda_s \geq 0$), by continuity in s the hypothesis $\frac{\lambda(M)}{\Lambda} := \delta_0 < 1$ implies $\frac{\lambda_s}{\Lambda} < \frac{1+2\delta_0}{3} < 1$ for s sufficiently close to p. And then we may choose $\delta > 0, \epsilon > 0$ sufficiently small that we have:

$$||w||_p^2 \leq \frac{2+\delta_0}{3}||w||_p^2 + C'_\epsilon ||w||_2^2, \text{ or } \frac{1-\delta_0}{3}||w||_p^2 < C'_\epsilon ||w||_2^2,$$

yielding an estimate $||w||_p^2 \leq C||w||_2^2$, which, as observed earlier, proves the lemma. □

From the lemma it is a short step to prove the main existence result of Trudinger and Aubin:

Theorem. Let ϕ_s be solutions to the subcritical Yamabe equation $\mathbb{Y}_s$, for $2 \leq s < p$, normalized to $||\phi_s||_s = 1$. Assume $\lambda(M) < \lambda(S^n) = \Lambda$. Then as $s \uparrow p$, a subsequence of $\{\phi_s\}$ converges uniformly on M to a smooth minimizer

$\phi > 0$ of the Yamabe quotient: $Q(\phi) = \lambda(M)$ and $||\phi||_p = 1$. In particular, the metric $\phi^{4/n-2}g$ has constant scalar curvature $\lambda(M)$ (and unit volume.)

It follows from this theorem that Yamabe's problem is solvable when $\lambda(M, [g]) \leq 0$ for the conformal class $[g]$.

Proof. By the lemma, $\{\phi_s\}$ is uniformly bounded in $L^r(M)$, for some $r > p$. As we saw in section 4, this implies $\{\phi_s\}$ is uniformly bounded in $C^{2,\alpha}$, for some $0 < \alpha < 1$. (Note that $p \geq (s-2)\frac{n}{2}$ is equivalent to $s \leq p$.) The Arzelà-Ascoli theorem then implies a subsequence ϕ_{s_i} converges to $\phi \in C^2$, uniformly in C^2 norm. The limit solves the limiting equation, namely the Yamabe equation with critical exponent:

$$L\phi = \lambda\phi^{p-1}, \quad \lambda := \lim_{s\uparrow p} \lambda_s.$$

By elliptic regularity, ϕ is smooth. Since $\phi \geq 0$, $\phi > 0$ follows from the maximum principle. We also have: $E(\phi_s) \to E(\phi)$ and $||\phi_s||_s = ||\phi||_p = 1$, so

$$Q_s(\phi_s) = \lambda_s(M) \to Q(\phi) = \lambda.$$

It remains to show that $\lambda = \lambda(M)$, and there are two cases to consider:

First, if $\lambda(M) \geq 0$. Then λ_s is decreasing in s and left-continuous at $s = p$. Hence in this case $\lambda = \lambda_p(M) = \lambda(M)$.

On the other hand, if $\lambda_p(M) < 0$, we have $\lambda_s < 0$ for $s \in [2, p]$, and is increasing (nondecreasing) in s. So we have $\lambda = \lim \lambda_s \leq \lambda(M)$. But $\lambda(M) \leq Q(\phi) = \lambda$; so also in this case, $\lambda = \lambda(M)$. □

6. Conformal blowup and a test function. To complete the solution of the Yamabe problem, we need to find a test function ϕ on M yielding the estimate $\lambda(M) < \lambda(S^n) = \Lambda$. The goal of [Lee-Parker] was to produce a unified argument, with appeal to the Positive Mass Theorem at the last step, in the cases where it applies. This is done in three steps: (1) Green's functions and the 'conformal blowup' construction; (2) Estimates for a test function; (3) The spherical defect rate and the ADM mass.

We consider point (3) first, starting with an observation regarding behavior of the ADM mass of AF manifolds under conformal change.

Proposition. (Exercise 3.12 in [Lee].) Suppose g is AF with rate q, and u is a positive function asymptotic to 1 also with rate q (both expressed in euclidean coordinates in the complement of a ball in R^n (r is the radial variable in R^n):

$$g_{ij(x)} = \delta_{ij} + O_2(r^{-q}), \quad u(x) = 1 + O_2(r^{-q}).$$

If the ADM mass is defined (say if $q > (n-2)/2$ and $S_g \in L^1(M, g)$) and $g' = u^{4/n-2}g$, we have:

$$m_{ADM}(g') = m_{ADM}(g) - \frac{2}{(n-2)\omega_{n-1}} \lim \oint_{S_r} (\partial_r u) d\omega_r^0.$$

As a corollary: if g is the euclidean metric and u has harmonic-type asymptotics:

$$u = 1 + \frac{A}{r^{n-2}} + O_2(r^{-n+1}),$$

Then $\partial_r u = -\frac{(n-2)A}{r^{n-1}} + O_1(r^{-n})$, and then: $m_{ADM}(u^{4/n-2}\delta) = 2A$. (Compare with the Schwarzschild example.)

Proof. For the mass integrand:

$$\partial_i g'_{ij} - \partial_j g'_{ii} = u^{p-2}[\frac{4}{n-2}(\frac{\partial_j u}{u} g_{ij} - \frac{\partial_j u}{h} g_{ii}) + (\partial_i g_{ij} - \partial_j g_{ii})].$$

And then straightforward calculation yields:

$$\partial_i g'_{ij} - \partial_j g'_{ii} = (1+O_2(r^{-q}))[-a\partial_j u + O(r^{-2q-1})] + (\partial_i g_{ij} - \partial_j g_{ii})], \quad a = \frac{4(n-1)}{n-2}$$

$$= -a\partial_j u + (\partial_i g_{ij} - \partial_j g_{ii}) + O(r^{-2q-1}).$$

Thus for the ADM mass:

$$m_{ADM}(g') = \frac{1}{2\omega_{n-1}} \lim \oint_{S_r} [-a\partial_j u + (\partial_i g_{ij} - \partial_j g_{ii})] \frac{x^j}{r} d\omega_r^0$$

$$= -\frac{2}{(n-2)\omega_{n-1}} \lim \oint_{S_r} (\partial_r u) d\omega_r^0 + m_{ADM}(g),$$

as claimed. □

In situations when the ADM mass is not defined, we can give a geometric interpretation of the integral of $\partial_r u$ over euclidean spheres, in terms of the derivative of the 'spherical area distortion function' $h(r)$:

$$h(r) = \frac{vol_g(S_r)}{\omega_r^0} = \frac{1}{\omega_r^0} \oint_{S_r} d\omega_r^g, \quad \omega_r^0 = \omega_{n-1} r^{n-1}.$$

We do this now for metrics (in an exterior euclidean domain) of the form:

$$g = u^{4/n-2}(\delta + O_2(r^{-2})), \quad u = 1 + Ar^{-k} + o_2(r^{-k}).$$

In particular:

$$\frac{1}{\omega_{n-1} r^{n-1}} \oint_{S_r} (\partial_r u) d\omega_r^0 = -kAr^{-k-1} + o_1(r^{-k-1}).$$

The goal now is to relate the coefficient A to the area function $h(r)$.

Proposition: Under the above conditions:

$$h(r) = 1 + \frac{a}{2} Ar^{-k} + o_2(r^{-k}), \quad h'(r) = -\frac{ka}{2} Ar^{-k-1} + o_1(r^{-k-1}), \quad a = \frac{4(n-1)}{n-2},$$

and hence:

$$\frac{1}{\omega_{n-1} r^{n-1}} \oint_{S_r} (\partial_r u) d\omega_r^0 = \frac{2}{a} h'(r) + o_1(r^{-k-1}).$$

Since the ADM mass (when defined) equals 2A, we see that *positive mass implies an eventually decreasing spherical area ratio.* Quantitatively: if $\sigma = \frac{a}{2}A$ is the first nontrivial coefficient in the expansion of $h(r)$, we have $m_{ADM} = \frac{n-2}{n-1}\sigma$.

Proof. The induced volume form on S_r is: $d\omega_r^g = N^g \lrcorner dV_g$, where N^gm the g-unit normal to S_r, is:

$$N^g = \frac{\nabla^g r}{|\nabla^g r|_g}, \quad \nabla^g r = g^{ij}\frac{x^j}{r}\partial_i, \quad |\nabla^g r|_g^2 = g^{kj}\frac{x^j x^k}{r^2},$$

since $dr = r^{-2}x^k dx^k$, metric-independent. Thus:

$$N^g = u^{-2/n-2}(\partial_r + O_2(r^{-2})),$$

where the error term is absent if g is conformally flat. Thus:

$$d\omega_r^g = N^g \lrcorner \sqrt{g} d^n x = u^{\frac{2n-2}{n-2}} d\omega_r^0 + O_2(r^{-2-k})$$

(where the error term is absent in the conf. flat case). From the expansion of u, this easily yields the desired expansion:

$$h(r) = \frac{1}{\omega_{n-1}r^{n-1}} \oint_{S_r} u^{\frac{2n-2}{n-2}} d\omega_r^0 + O_2(r^{-2-k}) = 1 + \frac{2n-2}{n-2}Ar^{-k} + o_2(r^{-k}).$$

$\square$

6.1. Green's function and the conformal blowup.

It is a classical fact that, on a compact Riemannian manifold, an elliptic operator of the form $L = -\Delta_g + h$ with h a positive function on M (so that the strong maximum principle holds for L) admits a positive Green's function $\Gamma = \Gamma_P$ (with pole at an arbitrary point $P \in M$), smooth in $M \backslash \{P\}$ with $L\Gamma = 0$ there, and satisfying $L\Gamma = \delta_P$ (Dirac mass at P) in the sense of distributions, that is:

$$\int_M \Gamma_P(x) Lf(x) dV_g(x) = f(P) \quad \forall f \in C^\infty(M).$$

In particular, letting $f \equiv 1$, we have the normalization: $\int_M \Gamma_P(x)h(x)dV_g(x) = 1$. The symmetry condition $\Gamma_P(x) = \Gamma_x(P)$ also holds.

It follows from this that the conformal Laplacian $L_g = -a\Delta_g + S_g$ also admits a Green's function, at least when $\Lambda(M) > 0$. Since S_g is not necessarily positive, this requires a short argument, given in [L-P, Lemma 6.1]. The proof uses existence for the subcritical Yamabe equation $\mathbb{Y}_s$ and the conformal invariance of L_g: if $\bar{g} = u^{4/n-2}g$ is a conformally related metric, we have:

$$L_{\bar{g}}(u^{-1}f) = u^{1-p}L_g f, \quad \forall f \in C^\infty(M).$$

It also uses the fact that Green's functions for the conformal Laplacian of metrics g and $\bar{g} = u^{4/n-2}g$ are related via:

$$\overline{\Gamma}_P(x) = u^{-1}(P)u^{-1}(x)\Gamma_P(x).$$

Recall that in R^n the Green's function Γ^0_P for the Laplacian Δ_0 with pole at P (satisfying $-\Delta_0\Gamma^0_P = \delta_P$) is given by:

$$\Gamma^0_P(x) = \frac{1}{(n-2)\omega_{n-1}}\frac{1}{r^{n-2}}, \quad r(x) = ||x-P||.$$

Thus $G^0_P = (n-2)\omega_{n-1}\Gamma^0_P = r^{2-n}$ satisfies $-\Delta_0 G^0_P = (n-2)\omega_{n-1}\delta_P$.

Setting $G_P = (n-2)\omega_{n-1}a\Gamma_P$ for the conformal Laplacian, we have:

$$-\Delta_g G_P = \frac{1}{a}L_g G_P - \frac{S_g}{a}G_P = (n-2)\omega_{n-1}\delta_P - (n-2)\omega_{n-1}S_g\Gamma_P,$$

in the distributional sense. Thus we 'expect' asymptotics with leading term r^{2-n} for G_P near P. In fact we have [L-P, Lemma 6.4]:

Lemma. In conformal normal coordinates at P, with $r = d(\cdot, P)$ the radial coordinate in a normal neighborhood U, we have the asymptotics as $r \downarrow 0$:

(i) If n=3,4,5 or g is conformally flat (CF) in U:

$$r^{n-2}G_P = 1 + Ar^{n-2} + O_2(r^{n-1}).$$

(ii) If $n = 6$:

$$r^{n-2}G_P = 1 - c_n|W(P)|^2 r^{n-2}\log r + O_2(r^{n-2}), \quad W = \text{Weyl tensor}.$$

(iii) If $n \geq 7$:

$$r^{n-2}G_P = 1 + [c_n|W(P)|^2 - \tilde{c}_n Hess(S)_{|P}(\partial_r, \partial_r)]r^4 + O_2(r^5).$$

We abbreviate these asymptotics a $r \to 0$ in the form (for $n \neq 6$):

$$r^{n-2}G_P = 1 + A(P)r^k + O_2(r^{k+1}); k = n-2, n = 3,4,5, CF; k = 4, n \geq 7. \quad (*)$$

Remark: Due to the logarithmic term, treatment of the case $n = 6$ is slightly different, but in inessential ways. The same results as for $n \geq 7$ hold for $n = 6$, but we have to omit it from the statements for this technical reason.

Conformal blowups. This construction associates to each compact manifold (M, g) (with $\lambda(M, [g]) > 0$) and point $P \in M$ a complete, non-compact one $(\hat{M}, \hat{g})$, with a single end. A neighborhood of the end in $\hat{M}$ corresponds to a punctured neighborhoord $U \setminus \{P\}$ of P in M, and $\hat{g}$ is asymptotically flat at its end. Let G_P be Green's function for the conformal Laplacian at P, normalized as above. We define:

$$(\hat{M}, \hat{g}) = (M \setminus \{P\}, \hat{g} = G_P^{4/n-2}g).$$

To check that $\hat{g}$ is conformally flat at its end, let (r, ω) be polar conformal normal coordinates in U, and let $M_\infty = U \setminus \{P\}$ represent the end of $\hat{M}$, identified via conformal normal coordinates with the complement of a ball in R^n. On M_∞

we introduce 'inverted polar conformal normal coordinates' $z = (\phi, \omega)$, where $z^i = r^{-2}x^i, \rho = |z| = r^{-1}, x^j = \rho^{-2}z^j$. Define $u(x) = r^{n-2}G_P(x)$ on $U \setminus \{P\}$. We have:

$$\hat{g} = G^{p-2}g = u^{p-2}r^{(2-n)(p-2)}g = u^{p-2}r^{-4}g = u^{p-2}\rho^4 g,$$

$$\hat{g}_{ij} = u^{p-2}g(\partial_{z_i}, \partial_{z_j}) = u^{p-2}(\delta_{ik} - 2\frac{z^i z^k}{\rho^2})(\delta_{jl} - 2\frac{z^j z^l}{\rho^2})g_{kl}(z),$$

while $g_{kl}(x) = \delta_{kl} + O_1(r^2)$ implies $g_{kl}(z) = \delta_{kl} + O_1(\rho^{-2})$. A short computation then yields:

$$\hat{g}_{ij}(z) = u^{p-2}(\delta_{ij} + O_1(\rho^{-2}))$$

(with the error term absent in the conformally flat case), where u has the abbreviated asymptotics given in (*) for $n \geq 6$, as $\rho \to \infty$:

$$u(z) = 1 + A(P)\rho^{-k} + O_2(\rho^{-k-1}), \quad k = k(n) \geq 1 \quad (***).$$

So we see $\hat{g}(z)$ is asymptotically flat on $\hat{M}_\infty$, of order $q = \min\{2, k\}$, with $k = 1, 2, 3$, $q = 1, 2, 2$ for $n = 3, 4, 5$ (resp.) and $k = 4$, hence $q = 2$ if $n \geq 7$; except in the case g conformally flat in U, when the order is $q = n - 2$ in all dimensions.

The metric $\hat{g}$ is scalar-flat (since $LgG_P = 0$ on $M \setminus \{P\}$), so the condition for the ADM mass to be defined is $q \geq (n-2)/2$. This holds in dimensions n=3,4 or 5, or in the CF case. So a PMT argument will apply only in those cases. On the other hand, the asymptotic expansion of the spherical volume ratio $h(\rho)$ (as given earlier) applies in all dimensions:

$$h(\rho) = 1 + \frac{a}{2}A\rho^{-k} + o_2(\rho^{-k}), \quad h'(\rho) = -\frac{ka}{2}A\rho^{-k-1} + o_1(\rho^{-k-1}) \quad (**)$$

and we have:

$$\frac{1}{\omega_{n-1}\rho^{n-1}} \oint_{S_\rho} (\partial_\rho u) d\omega_\rho^0 = \frac{2}{a}h'(\rho) + o_1(\rho^{-k-1}).$$

6.2 Definition and estimates for a test function.

Let $u_\alpha(|z|) = (\frac{\alpha}{\alpha^2+\rho^2})^{\frac{n-2}{2}}$ $(\alpha > 0)$ be the one-parameter family of minimizers for S^n (with its standard conformal structure), transferred to R^n via stereographic projection. Recall :

$$-a\Delta_0 u_\alpha = 4n(n-1)u_\alpha^{p-1}, \quad \Lambda = \lambda(S^n) = 4n(n-1)||u_\alpha||_p^{p-2}.$$

Fix $R >> 1$ large, and let $\hat{M}_R = \{z \in \hat{M}_\infty; \rho(z) \geq R\}$. Extend ρ to all of $\hat{M}$ as a positive smooth function. The test function $\phi : \hat{M} \to R_+$ is defined on $\hat{M}_\infty$ as:

$$\phi(z) = u_\alpha(|z|), \rho(z) \geq R; \quad \phi(z) = u_\alpha(R), \rho(z) \leq R,$$

and extend ϕ to the rest of $\hat{M}$ as the constant $u_\alpha(R)$. ϕ is not quite smooth, only Lipschitz (or in $W^{1,2}$), but this is good enough. Note $\phi(z) \to 0$ at infinity with all its derivatives; so we may regard ϕ as a smooth function on M, with $\phi(P) = 0$ and ϕ constant outside a small neighborhood of P. The strategy will be to take R large, and then choose $\alpha >> R$.

Underlying this argument is the following easily verified fact: $\lambda(M, [g]) = \lambda(\hat{M}, \hat{g})$.

Proposition. For $n \neq 6$, or g conformally flat in U, we have:

$$E(\phi) \leq \Lambda ||\phi||_p^2 - C_n \mu \alpha^{-k} + O(\alpha^{-k-1}).$$

Here C_n depends only on dimension and $\mu = kA$ (for the $A = A(P)$ in (***)), while $k = n - 2$ for $n = 3, 4, 5, CF$ and $k = 4$ if $n \geq 7$.

Proof. Here $E(\phi)$ is computed with respect to the metric $\hat{g}$ in $\hat{M}$. Since $\hat{g}$ is scalar-flat on $\hat{M}$,

$$E(\phi) = \int_{\hat{M}} a|\nabla\phi|^2_{\hat{g}} dV_{\hat{g}} = \int_{\hat{M}_\infty} a\hat{g}^{\rho\rho}(\partial_\rho u_\alpha)^2 u^p dV_g = a\int_{\hat{M}_{R/2}} (\partial_\rho u_\alpha)^2 u^2 d^n z.$$

Here the following facts were used (from left to right): ϕ is constant outside the end $\hat{M}_\infty$; $\hat{g} = u^{p-2}g$, so $dV_{\hat{g}} = u^p dV_g$ and $\hat{g}^{\rho\rho}(z) = u^{-(p-2)}$, since $g^{rr}(x) = 1$ (with no error terms) in normal coordinates; finally, $dV_g(z) = d^n z$ up to an arbitrarily small error term (neglected in the estimate), since we adopted conformal normal coordinates in U.

Next we recall $(\partial_\rho u_\alpha)^2 = |\nabla^0 u_\alpha|^2_0$ and integrate by parts in the euclidean metric. The domain of integration is an n-dimensional euclidean annulus $A(R, L) = \{z \in \hat{M}_\infty; R \leq |z| \leq L\}$, where we'll soon let $L \to \infty$.

$$\int_{A(R,L)} |\nabla_0 u_\alpha|_0^2 u^2 d^n z = -\int_{A(R,L)} u_\alpha(\Delta)_0 u^2 d^n z - \int_{A(R,L)} u_\alpha \partial_\rho u_\alpha(\partial_\rho u^2) d^n z - \int_{S_R \sqcup S_L} u_\alpha(\partial_\rho u_\alpha) u^2 d\sigma_0.$$

On S_L we use $|\partial_\rho u_\alpha| \leq (n-2)\alpha^{\frac{n-2}{2}}\rho^{1-n}, |u_\alpha| \leq \alpha^{\frac{n-2}{2}}\rho^{2-n}$, so $u_\alpha|\partial_\rho u_\alpha|$ is bounded above on S_L by $(n-2)\alpha^{n-2}L^{3-2n}$, which integrated over S_L is $O(L^{2-n})$, as $L \to \infty$.

On S_R we use $|\partial_\rho u_\alpha| \leq (n-2)\alpha^{\frac{n-2}{2}} R$ and $|u_\alpha| \leq \alpha^{-\frac{n-2}{2}}$, so their product integrated over S_R is $O(R^n \alpha^{-n})$, which vanishes if we take R large first, then let $\alpha \to \infty$. Thus the boundary terms don't contribute.

$$-\int_{A(R,L} u_\alpha(\Delta)_0 u^2 d^n z = 4n(n-1)\int_{A(R,L)} u_\alpha^{p-2}(u_\alpha u)^2 d^n z,$$

and this can be estimated via Hölder's inequality, applied to $f = u_\alpha^{p-2}$ (in $L^q, q = \frac{p}{p-2}$), and $g = (u_\alpha u)^2$ (in $L^{q'}, q' = \frac{p}{2}$). We find:

$$\int_A fg d^n z \leq ||f||_q ||g||_{q'} = ||u_\alpha||^{p-2}_{L^p(A)} (\int_A \phi^p dV_g)^{2/p} \leq \Lambda ||\phi||^2_{L^p(A)}.$$

Letting $L \to \infty$, we conclude:

$$E(\phi) \leq \Lambda ||\phi||_p^2 - \int_{A[R,\infty)} a u_\alpha (\partial_\rho u_\alpha)(\partial_\rho u^2) d^n z + O_R(\alpha^{-n}).$$

It remains to estimate the crucial term. Recall that under the asymptotics $(***)$ for u, we have (for $n \neq 6$):

Lemma A:

$$\frac{1}{\omega_\rho} \oint_{S_\rho} (\partial_\rho u) d\omega_\rho^0 = -kA\rho^{-k-1} + O_1(\rho^{-k-2}), \quad \omega_\rho = \omega_{n-1}\rho^{n-1}.$$

Thus we have the estimate:

$$a \int_{S_\rho} \partial_\rho u^2 d\omega_\rho^0 = 2a \int_{S_\rho} (1+O(\rho^{-k}))(\partial_\rho u) d\omega_\rho^0 = 2a\omega_\rho [\frac{1}{\omega_\rho} \oint_{S_\rho} (\partial_\rho u) d\omega_\rho^0 + O(\rho^{-2k-1})]$$

$$= 2a\omega_\rho[-kA\rho^{-k-1} + O(\rho^{-k-2})],$$

using Lemma A. Recalling that $u_\alpha \partial_\rho u_\alpha = -(n-2)\rho\alpha^{-1}(\frac{\alpha}{\alpha^2+\rho^2})^{n-1}$, we need a 'calculus lemma':

Lemma B:

$$\frac{1}{C}\alpha^{-k+1} \leq \int_R^\infty \rho^{-k}(\frac{\alpha}{\alpha^2+\rho^2})^{n-1}\rho^{n-1} d\rho \leq C\alpha^{-k+1},$$

where $C > 1$ depends only on n and R.

Thus we have the estimate:

$$-a \int_R^\infty u_\alpha (\partial_\rho u_\alpha)(\int_{S_\rho} \partial_\rho u^2 d\omega_\rho^0) d\rho$$

$$= (n-2)\int_R^\infty \rho\alpha^{-1}(\frac{\alpha}{\alpha^2+\rho^2})^{n-1} 2a\omega_\rho[-kA\rho^{-k-1} + O(\rho^{-k-2})] d\rho$$

$$= -2a\omega_{n-1}(n-2)kA\alpha^{-1}\int_R^\infty \rho^{-k}(\frac{\alpha}{\alpha^2+\rho^2})^{n-1}\rho^{n-1} d\rho + C\alpha^{-1}\int_R^\infty \rho^{-k-1}(\frac{\alpha}{\alpha^2+\rho^2})^{n-1}\rho^{n-1} d\rho$$

$$= -c_n kA\alpha^{-k} + O(\alpha^{-k-1}),$$

appealing to Lemma B twice. Thus we have:

$$E(\phi) \leq \Lambda ||\phi||_p^2 - c_n kA\alpha^{-k} + O(\alpha^{-k-1}),$$

establishing the proposition with $\mu = kA$. □

Conclusion of the solution of Yamabe's problem. To establish that $\lambda(M) < \Lambda$ (if $(M, [g])$ is not conformally equivalent to the sphere), we have two cases.

Case (i): $n = 3, 4, 5$ or g conformally flat in U. Then the conformal blowup $(\hat{M}, \hat{g})$ is an asymptotically flat manifold satisfying the conditions that make the ADM mass well-defined. Since we are under conditions where the positive mass theorem holds (recall $S_{\hat{g}} \equiv 0$), and since we earlier established that $m_{ADM} = 2A$, it follows that $A > 0$ (hence $\mu > 0$), or else $(\hat{M}, \hat{g})$ is isometric to R^n with the flat metric, which is only possible if $(M, [g])$ is the sphere with its standard conformal class. And if $A > 0$, we can certainly take R large enough, then α large enough, so that $E(\phi) < \Lambda ||\phi||_p^2$, leading to $\lambda(M) < \Lambda$.

Case (ii): If $n \geq 6$ and M is not locally conformally flat, the conclusion $A > 0$ follows by picking a point P where the Weyl tensor does not vanish, and recalling the precise expression for A (more details are needed here; in any case, these cases were resolved in [Aubin 76]). Alternatively, if we know the spherical area ratio function $h(r)$ (for the conformal blowup) is eventually decreasing, then it also follows that $A > 0$.